A Lean proof guide

Conway's refinement conjecture for omnific integers

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Overview

Conway's refinement conjecture asks whether every equality of two products of omnific integers admits a four-factor refinement. For this domain, that is equivalent to the pre-Schreier property: every element is primal. Signed Conway normal form identifies the ring with a bounded generalised-power-series integer part.

Over a characteristic-zero field, Cantor–Bendixson analysis and translated truncations prove algebraic independence of minimal homogeneous generators; generation then gives a polynomial presentation. The finite-support coefficient ring and its multivariate polynomial rings have greatest common divisors; the presentation transfers them to the real-exponent series ring. Under the general theorem's finite-class and common-tail hypotheses, primality of a bounded integer part over an ordered rational vector space follows by induction on the order type of the Archimedean classes of the non-zero exponents in the support. At infinite rank, the common-tail quotient gives an exact refinement; factoring off a divisor supported in the corresponding closed Archimedean ball leaves a cofactor of smaller rank. For surreal exponents, cut filling, coinitial quotient scales, and rational halving give quotient completeness, while cut-proved common-tail cofinality gives the fraction-field condition. The transfinite theorem then applies, and signed normal form transports refinement back to omnific integers.

0. Formal statements

Three standalone files make the result easier to audit.

- Conway's refinement conjecture using the surreal numbers from CombinatorialGames;
- a Mathlib-only construction of the needed surreal numbers, order, and arithmetic;
- a Mathlib-only refinement theorem for integer parts of generalised power-series fields.

0.1 Conway's cut formulation

The concise *CombinatorialGames statement* uses Conway's equation

$$x = \{x - 1 \mid x + 1\}$$

to define omnific integers. If a, b, c, d satisfy this equation and $ab = cd$, then there are omnific integers e, f, g, h such that

$$a = ef, \qquad b = gh, \qquad c = eg, \qquad d = fh.$$

0.2 Conway's refinement conjecture from first principles

The *Mathlib-only statement* constructs well-founded games, Conway's arithmetic and order, numeric games, surreal-number representatives, the omnific-integer cut, and the same four-factor proposition in one file.

0.3 Refinement over saturated exponent groups

The *generalised-power-series statement* works entirely in Mathlib. For a regular uncountable cardinal κ, a κ-saturated ordered rational vector space G, and a characteristic-zero field R, every equality $ab = cd$ in the cardinal-bounded integer part

$$\mathbb{Z} + R((G^{<0}))_\kappa$$

admits the same four-factor refinement inside that integer part.

1 Algebraic and ordinal preliminaries

Before the proof can work with series, it needs two sets of tools. The algebraic tools cover divisibility and greatest common divisors, and describe polynomial relations in graded rings. They also connect a filtered ring to its associated graded ring. The ordinal tools describe Cantor normal forms and Hessenberg's natural sum, especially what can happen when one or more summands become smaller. These results do not yet form one argument; later chapters use them at several different points.

The two sets of tools meet in the degree induction. If a family of generators satisfied a polynomial relation, we will study its partial derivatives and the homogeneous ideal it generates. At the same time, we will use ordinal bounds to compare the terms produced by translated truncations. The final lifting result in this chapter will later help turn generation of an associated graded ring into generation of the original algebra.

Lemma 1 (A primal zero-residue element forces the fraction-field condition)**.** *Let L be a field, let A be a domain and an L-algebra, and let $\pi : A \to L$ be an L-algebra retraction. Let $S \subseteq L$ be a subring. Suppose that $p \in A$ is nonzero, $\pi(p) = 0$, and every factorisation $p = ab$ has $\pi(a) \neq 0$ or $\pi(b) \neq 0$. If p is primal in $\pi^{-1}(S)$, then L is the fraction field of S.*

Proof. Fix $x \in L^{\times}$. In $\pi^{-1}(S)$, primality of p applied to

$$(xp)(x^{-1}p) = p^2$$

gives a factorisation $p = p_1p_2$ with $p_1 \mid xp$ and $p_2 \mid x^{-1}p$. At least one of $\pi(p_1)$ and $\pi(p_2)$ is nonzero. Cancelling p from the corresponding divisibility equation expresses x as a quotient of two nonzero residues of elements of $\pi^{-1}(S)$. Both residues lie in S, so $x \in \mathrm{Frac}(S)$. The case $x = 0$ is immediate. □

`Subring.fracSubring_eq_top_of_isPrimal_of_residue_eq_zero`

Lemma 2 (Extension of a degreewise independent family)**.** *Let E be a field, let R be a commutative E-algebra, and let $(A_\alpha)_{\alpha\in\mathbf{On}}$ be a family of E-subspaces of R. Put*

$$D_\beta = \sum_{\substack{i\oplus j=\beta \\ i,j\neq 0}} A_iA_j.$$

Suppose $w_i \neq 0$, $x_i \in A_{w_i}$, and every finitely supported combination $\sum_i c_ix_i$ with all $w_i = \beta$ belongs to D_β only when every c_i is zero. Then there are a family $(x'_j)_{j\in I'}$, weights w'_j, and an injection $e : I \to I'$ such that $w'_j \neq 0$, $x'_j \in A_{w'_j}$, $w'_{e(i)} = w_i$, and $x'_{e(i)} = x_i$. Moreover, the x'_j of weight β are independent modulo D_β for every β and span A_β modulo D_β whenever $\beta \neq 0$.

Proof. For every β, pass to $A_\beta/(A_\beta \cap D_\beta)$. Extend the prescribed independent classes to a basis and choose lifts in A_β for the added basis vectors. The union over all nonzero weights contains the original family and its classes give the required independence and spanning properties weight by weight. □

`OrdinalGraded.exists_isMinimalSystem_extension`

Lemma 3 (Homogeneous decomposition in a finitely generated graded ideal)**.** *For a finite family of homogeneous elements $Q_j \in A_{c_j}$, if $x \in A_\alpha$ lies in the ideal generated by the Q_j, then there are u_j such that $x = \sum_j Q_ju_j$, where $u_j \in A_\beta$ whenever $\beta \oplus c_j = \alpha$, and $u_j = 0$ if no such β exists.*

Proof. Apply the componentwise ideal representation and use $x_\alpha = x$. $\square$

`OrdinalGraded.exists_eq_sum_mul_of_mem_span`

Lemma 4 (Weighted-homogeneous polynomial representatives)**.** *Let E be a field, let R be a commutative E-algebra, and let $(A_\alpha)_{\alpha \in \mathbf{On}}$ be a family of E-subspaces of R. Put*

$$D_\beta = \sum_{\substack{i \oplus j = \beta \\ i,j \neq 0}} A_i A_j.$$

Let (x_i, w_i) have positive weights, with $x_i \in A_{w_i}$. Suppose the x_i of weight β are independent modulo D_β for every β and span A_β modulo D_β whenever $\beta \neq 0$. If every element of A_0 is a scalar from E, then every $y \in A_\beta$ equals $F(x)$ for some $F \in E[X_i : i \in I]$ weighted-homogeneous of weight β.

Proof. Proceed by well-founded induction on β. At weight zero the hypothesis makes y a constant polynomial value. At positive weight, write y modulo D_β as a linear combination of the weight-β generators. Every product defining D_β has two positive weights strictly below β, so induction represents both factors by weighted-homogeneous polynomials. Adding their products to the linear combination gives the required polynomial. $\square$

`OrdinalGraded.IsMinimalSystem.exists_aeval_eq`

Lemma 5 (Linear part of a weighted-homogeneous polynomial)**.** *Let E be a field and let $A = \bigoplus_{\alpha \in \mathbf{On}} A_\alpha$ be a commutative E-algebra graded by the ordinals in a fixed universe, with multiplication graded by Hessenberg sum. Let $(x_i)_{i \in I}$ satisfy $x_i \in A_{w(i)}$ with $w(i) \neq 0$. If $F \in E[X_i : i \in I]$ is weighted-homogeneous of degree $\beta \neq 0$, then*

$$F(x) \equiv \sum_{w(i) = \beta} c_i x_i \bmod (A_+)^2 \cap A_\beta,$$

where c_i is the coefficient of X_i in F.

Proof. It suffices to inspect one monomial. A monomial cX_i of degree β contributes its linear term, while every other nonconstant monomial factors into two positive-degree monomials and therefore evaluates into $(A_+)^2 \cap A_\beta$. $\square$

`OrdinalGraded.exists_linear_part`

Lemma 6 (Vanishing of the linear part of a homogeneous relation)**.** *If $(x_i)_{i \in I}$ is a minimal system of homogeneous generators and the weighted-homogeneous polynomial F of nonzero degree β satisfies $F(x) = 0$, then the coefficient of every linear monomial X_i in F is zero.*

Proof. Lemma 5 (*Linear part of a weighted-homogeneous polynomial*) places the linear combination of the degree-β generators in $(A_+)^2 \cap A_\beta$. Their defining independence modulo this subspace kills every coefficient. $\square$

`OrdinalGraded.IsMinimalSystem.coeff_single_eq_zero_of_aeval_eq_zero`

Lemma 7 (Weights of variables in a homogeneous relation)**.** *Every variable occurring in a nonzero-degree weighted-homogeneous relation among a minimal system $(x_i)_{i\in I}$ of homogeneous generators has weight strictly below the degree of the relation.*

Proof. Weighted homogeneity gives $w(i) \le \beta$. Equality would make the monomial containing X_i linear, contradicting Lemma 6 (*Vanishing of the linear part of a homogeneous relation*). ☐

`OrdinalGraded.IsMinimalSystem.wt_lt_of_mem_vars_of_aeval_eq_zero`

Lemma 8 (Weighted Euler decomposition at the ω^0 coefficient)**.** *Let F be weighted homogeneous of degree δ, and let $n > 0$ be the coefficient of $1 = \omega^0$ in the Cantor normal form of δ. Then*

$$F_0 + \sum_{\substack{i\in\mathrm{vars}(F)\\ \mathrm{coeff}_{\omega^0}(\mathrm{wt}(i))=n}} X_i \frac{\partial F}{\partial X_i} = F,$$

where F_0 is obtained by setting precisely those variables to zero.

Proof. Expand F into monomials. If a monomial contains a variable whose weight has constant Cantor coefficient n, additivity of that coefficient under Hessenberg sum shows that the variable occurs exactly once and all other variable weights have constant Cantor coefficient zero. Otherwise the monomial is fixed by the substitution. Summing over the monomials gives the formula. ☐

`MvPolynomial.eraseFinitePart_add_sum_X_mul_pderiv`

Lemma 9 (Partial-derivative decomposition of a homogeneous ideal relation)**.** *Let F be weighted homogeneous of degree δ, and let $n > 0$ be the coefficient of $1 = \omega^0$ in the Cantor normal form of δ. Suppose*

$$F \in \big(\partial_i F : i \in \mathrm{vars}(F),\ \mathrm{coeff}_{\omega^0}(\mathrm{wt}(i)) = n\big).$$

Then there are weighted-homogeneous polynomials V_i of degree $\mathrm{wt}(i)$*, involving no variable whose weight has constant Cantor coefficient n, such that*

$$F = \sum_{\substack{i\in\mathrm{vars}(F)\\ \mathrm{coeff}_{\omega^0}(\mathrm{wt}(i))=n}} \frac{\partial F}{\partial X_i}(X_i + V_i).$$

Proof. By Lemma 3 (*Homogeneous decomposition in a finitely generated graded ideal*), homogeneous ideal membership writes $F = \sum_i (\partial_i F)U_i$, with U_i weighted homogeneous of degree $\mathrm{wt}(i)$. Let V_i be obtained from U_i by setting to zero every variable whose weight has constant Cantor coefficient n. Applying the same substitution to the displayed ideal identity and combining it with Lemma 8 (*Weighted Euler decomposition at the ω^0 coefficient*) gives the required formula. ☐

`MvPolynomial.exists_eq_sum_pderiv_mul_X_add_of_mem_span`

Lemma 10 (Greatest common divisors in multivariate polynomial rings)**.** *Let R be a commutative domain admitting a normalized GCD structure, and let I be any type. Then $R[X_i : i \in I]$ admits greatest common divisors.*

Proof. A pair of polynomials involves only finitely many variables. Regard their finite-variable ring as an iterated univariate polynomial ring and choose a greatest common divisor there. View the full ring as the polynomial ring in the complementary variables over this finite-variable ring. If at least one of the original polynomials is non-zero, a common divisor of their constant images is associated to a constant polynomial; its coefficient divides both finite-variable polynomials and hence their greatest common divisor. The case where both polynomials vanish is immediate. Thus the finite-variable greatest common divisor remains one in the full polynomial ring. □

`MvPolynomial.nonemptyGCDMonoid`

Lemma 11 (Partial-derivative obstruction to a linear variable). *For an ordinal ξ, let $\xi_{<\beta}$ be the Hessenberg sum of the terms in its Cantor normal form whose exponents are below β.*

Let R be a commutative ring of characteristic zero without zero divisors. Give the variables X_i ordinal weights w_i, and let $F \in R[X_i : i \in I]$ be weighted homogeneous of degree α. Suppose that X_{B_0} occurs in F with degree at most one and $(w_{B_0})_{<\beta} = \alpha_{<\beta}$. Put $H = \partial_{B_0} F$, and suppose that $H \neq 0$ is weighted homogeneous of some nonzero degree δ.

Fix an ordinal λ_0. Suppose that for every variable $X_{B'}$ occurring in F such that $(w_{B'})_{<\beta} \oplus \eta = \lambda_0$ for some ordinal η, there are a finite set S and polynomials U_B satisfying

$$(w_B)_{<\beta} \neq 0, \qquad \partial_{B_0} U_B = 0 \quad (B \in S), \qquad \partial_{B'} F = \sum_{B \in S} (\partial_B F) U_B.$$

These hypotheses are inconsistent.

Proof. Choose a variable $X_{B'}$ occurring in the nonzero homogeneous polynomial H. Its degree is nonzero, so H has a nonconstant monomial and B' can be chosen with $\partial_{B'} H \neq 0$. Because X_{B_0} occurs at most once and already accounts for every Cantor term below ω^β in the degree of F, every variable of H has $(w_{B'})_{<\beta} = 0$. The equation $0 \oplus \lambda_0 = \lambda_0$ therefore supplies the assumed identity for $\partial_{B'} F$.

Differentiate this identity with respect to X_{B_0}. The left side is $\partial_{B'} H \neq 0$. On the right, every U_B is independent of X_{B_0}, while $(w_B)_{<\beta} \neq 0$ prevents X_B from occurring in H; the Leibniz rule therefore makes every summand zero, a contradiction. □

`MvPolynomial.false_of_pderiv_eq_sum_of_partLT_ne_zero`

Lemma 12 (Vanishing criterion for partial derivatives of weighted-homogeneous polynomials). *If F is weighted-homogeneous of degree δ and $\delta \neq \beta \oplus w(i)$ for every β, then $\partial_i F = 0$.*

Proof. Any monomial containing X_i would express δ as the natural sum of $w(i)$ and the weight left after removing one occurrence of X_i. □

`MvPolynomial.pderiv_eq_zero_of_isWeightedHomogeneous`

Lemma 13 (Weighted degree of polynomial derivations). *Suppose g_i is weighted-homogeneous of the degree obtained from $w(i)$ by subtracting one from its positive constant Cantor coefficient, and is zero when that coefficient vanishes. If F is weighted-homogeneous of degree δ, then $D_g(F)$ is weighted-homogeneous of the degree obtained by truncated subtraction of one from the constant Cantor coefficient of δ (so the degree remains δ when that coefficient is zero).*

Proof. For a monomial, $\partial_i F$ removes $w(i)$ from the weight, while multiplication by g_i restores it with one subtracted from its constant Cantor coefficient. Hessenberg addition therefore subtracts one from the constant Cantor coefficient of δ. Linearity handles sums. □

`MvPolynomial.mkDerivation_isWeightedHomogeneous_removeNat`

Lemma 14 (Weighted Euler identity)**.** *If F is weighted-homogeneous of degree δ and S is any finite set containing its variables, then*

$$\sum_{i \in S} n_i X_i \frac{\partial F}{\partial X_i} = nF,$$

where n_i and n are the constant Cantor coefficients of $w(i)$ and δ.

Proof. On a monomial, the coefficient contributed by $X_i \partial_i$ is the multiplicity of X_i. Weighted by the constant Cantor coefficient of $w(i)$ and summed over i, this is the constant Cantor coefficient of the monomial's total weight, namely n. Extend by linearity. □

`MvPolynomial.IsWeightedHomogeneous.sum_constantCoeff_X_mul_pderiv`

Lemma 15 (Finite generation of polynomial syzygies)**.** *Let B be a finite set, let $\Lambda \subseteq I$ be finite, and let $c_b \in K[X_i : i \in \Lambda]$ for every $b \in B$. There is a finite family of tuples $\sigma \in K[X_i : i \in \Lambda]^B$ satisfying $\sum_{b \in B} c_b \sigma_b = 0$ that generates, over $K[X_i : i \in I]$, every tuple $u \in K[X_i : i \in I]^B$ satisfying $\sum_{b \in B} c_b u_b = 0$.*

Proof. The ring $K[X_i : i \in \Lambda]$ is Noetherian, so the kernel of $(u_b) \mapsto \sum_b c_b u_b$ is finitely generated. Identify $K[X_i : i \in I]$ with a polynomial ring in the remaining variables over $K[X_i : i \in \Lambda]$. Expanding an arbitrary syzygy in those variables makes each coefficient tuple a syzygy over $K[X_i : i \in \Lambda]$. Express each such tuple in the chosen finite generating family and reassemble the polynomial. □

`MvPolynomial.exists_finset_syzygy_span_supported`

Lemma 16 (Lifting generation from the associated graded ring)**.** *Let ν be a separated multiplicative degree on a ring R, with well-founded value order, and let $P \subseteq R$ be a subalgebra. If every element of $\operatorname{gr}_\nu R$ is a finite sum of initial forms of elements of P, then $P = R$.*

Proof. Use well-founded induction on $\nu(t)$. Express the initial form of t as a finite sum of initial forms of elements of P, and subtract their sum $t' \in P$. Equality of initial forms gives $\nu(t - t') < \nu(t)$, so the induction hypothesis puts $t - t'$ in P. Hence $t = (t - t') + t'$ lies in P. Separatedness handles degree $\perp$. □

`MaxAddDegree.mem_of_forall_exists_sum_initialForm`

Lemma 17 (Separation below the last Cantor term)**.** *Let $\sigma \neq 0$, let ω^β be its last Cantor term, and write $\rho \oplus \sigma = h$. If the sum of the terms in the Cantor normal form of h at exponents at least β is at most τ, then $\rho \oplus \theta < \tau$ for every $\theta < \sigma$.*

Proof. Split each Cantor normal form at exponent β. For $\theta < \sigma$, the terms of θ at exponents at least β, followed by one further ω^β, are bounded by σ. Hence $\rho \oplus \theta$ is strictly below the assumed bound for the terms of $\rho \oplus \sigma$ at those exponents, and therefore below τ. □

`NatOrdinal.add_lt_of_lt_of_partGE_le`

Lemma 18 (Well-ordering of a countable ordered union)**.** *Let (Y_n) be well-ordered subsets of a linear order, and suppose $x < y$ whenever $x \in Y_j$, $y \in Y_k$, and $j < k$. Then $\bigcup_n Y_n$ is well ordered.*

Proof. A decreasing sequence in the union determines a nonincreasing sequence of block indices, hence eventually lies in one Y_n, contradicting that Y_n is well ordered. ☐

`Set.IsPWO.iUnion_of_ordered`

Lemma 19 (Order type of a countable ordered union)**.** *Let (Y_n) be well-ordered subsets of a linear order, and suppose $x < y$ whenever $x \in Y_j$, $y \in Y_k$, and $j < k$. If $\mathrm{ot}(Y_n) \leq \rho$ for every n, and $n \odot \rho < o$ for every $n \in \mathbb{N}$, then $\mathrm{ot}(\bigcup_n Y_n) \leq o$.*

Proof. By Lemma 18 (*Well-ordering of a countable ordered union*), the union is well ordered. Every initial segment ending in Y_n lies in the finite union of the first $n + 1$ blocks, whose order type is at most the Hessenberg sum of $n + 1$ copies of ρ. ☐

`Set.IsPWO.orderType_iUnion_le_of_ordered`

Lemma 20 (Countable ordered unions below ω^e)**.** *Let (Y_n) be well-ordered subsets of a linear order, and suppose $x < y$ whenever $x \in Y_j$, $y \in Y_k$, and $j < k$. If $\mathrm{ot}(Y_n) < \omega^e$ for every n, then $\mathrm{ot}(\bigcup_n Y_n) \leq \omega^e$.*

Proof. By Lemma 18 (*Well-ordering of a countable ordered union*), the union is well ordered. Induction on m shows that the union of the first m blocks has order type below ω^e: the order type of a union is bounded by the Hessenberg sum, and the Hessenberg sum of two ordinals below ω^e remains below ω^e. Every initial segment of the full union is contained in one such finite union. ☐

`Set.IsPWO.orderType_iUnion_le_wpow_of_ordered`

Lemma 21 (Cofinality below a Hessenberg sum)**.** *Let $\lambda \neq 0$ and σ be ordinals such that $\sigma = 0$ or every term of the Cantor normal form of σ is at least the last term of the Cantor normal form of λ. Then for every $\tau < \lambda \oplus \sigma$ there is $\rho < \lambda$ with $\tau \leq \rho \oplus \sigma$.*

Proof. Write $\lambda = \lambda' \oplus \omega^e$, where ω^e is its last Cantor-normal-form term. The hypothesis on σ gives $\lambda \oplus \sigma = (\lambda' \oplus \sigma) + \omega^e$. If τ lies below $\lambda' \oplus \sigma$, take $\rho = \lambda'$. Otherwise write $\tau = (\lambda' \oplus \sigma) + \nu$ with $\nu < \omega^e$ and take $\rho = \lambda' \oplus \nu$; strict monotonicity gives $\rho < \lambda$, and ordinary addition is bounded by the natural sum. ☐

`NatOrdinal.exists_lt_le_add_of_lastCantorTerm_le`

Lemma 22 (Uniform bound for simultaneous decreases in a Hessenberg sum)**.** *Let $\sigma_1, \sigma_2 \neq 0$, with last Cantor terms ω^{e_1} and ω^{e_2}. There is $\mu' < \rho \oplus \sigma_1 \oplus \sigma_2$ such that*

$$\rho \oplus \theta_1 \oplus \theta_2 \leq \mu'$$

for all $\theta_1 < \sigma_1$ and $\theta_2 < \sigma_2$.

Proof. Interchange the indices if necessary so that $e_1 \le e_2$. If $e_1 = e_2$, retain the terms of ρ at exponents at least e_1, remove the last term from each σ_i, and retain one copy of ω^{e_1}. If $e_1 < e_2$, retain the terms of $\rho \oplus \sigma_1 \oplus \sigma_2$ at exponents at least e_2. The resulting ordinal is strictly below the original sum and bounds every pair of proper lowerings. □

`NatOrdinal.exists_lt_forall_add_add_le`

Lemma 23 (Weighted-degree bound for terms with two translated truncations)**.** *For an ordinal ξ, write $\xi_{<\beta}$ and $\xi_{\ge\beta}$ for the Hessenberg sums of the terms in its Cantor normal form whose exponents are, respectively, below β and at least β.*

Let K be a commutative ring, give each variable X_i an ordinal weight w_i, and let $F \in K[X_i : i \in I]$ be weighted homogeneous of degree α, with $\alpha_{<\beta} \ne 0$. Suppose that whenever X_i occurs in F and $(w_i)_{<\beta} \ne 0$, the last term of the Cantor normal form of $(w_i)_{<\beta}$ is ω^e for some $e \ne 0$.

There is $\lambda < \alpha_{<\beta}$ such that the following holds. For every monomial X^d occurring in F, every factorisation $X^d = X^{d'} X_{i_1} \cdots X_{i_k}$ with $k \ge 2$, and all $\rho_j < w_{i_j}$, put $\rho = w(d') \oplus \rho_1 \oplus \cdots \oplus \rho_k$, where $w(d')$ is the weighted degree of $X^{d'}$. If $\rho_{\ge\beta} = \alpha_{\ge\beta}$, then $\rho_{<\beta} \le \lambda$.

Proof. For each ordered pair of variables occurring in F, Lemma 22 (*Uniform bound for simultaneous decreases in a Hessenberg sum*) gives an ordinal below $\alpha_{<\beta}$ that bounds every simultaneous proper lowering of that pair; take the maximum of these finitely many bounds. A term with at least two translated truncations contains such a pair. Equality of its part at or above β with $\alpha_{\ge\beta}$ forces both selected factors to retain their parts at or above β. Its part below β is therefore bounded by the chosen pair bound, hence by the finite maximum. □

`MvPolynomial.exists_forall_partLT_le_of_termDegree`

Lemma 24 (Rigidity of upper Cantor terms under simultaneous decreases)**.** *Suppose $\sigma_i \ne 0$ have last Cantor terms ω^{e_i} with $e_1 < e_2$. Let $\mu' < \rho \oplus \sigma_1 \oplus \sigma_2$ bound every $\rho \oplus \theta_1 \oplus \theta_2$ with $\theta_i < \sigma_i$. Then μ' and $\rho \oplus \sigma_1 \oplus \sigma_2$ have the same terms in their Cantor normal forms at exponents at least e_2.*

Proof. If these terms differed, remove the last Cantor term from each σ_i and replace the last term of σ_2 by a sufficiently large smaller ordinal. This gives a permitted pair of proper lowerings whose sum exceeds μ', contradicting the assumed bound. □

`NatOrdinal.partGE_eq_of_forall_add_add_le`

Lemma 25 (Hessenberg-sum decomposition with unequal last Cantor terms)**.** *Suppose $\sigma_i \ne 0$ have last Cantor terms ω^{e_i} with $e_1 < e_2$, and $\mu' < \rho \oplus \sigma_1 \oplus \sigma_2$ bounds every $\rho \oplus \theta_1 \oplus \theta_2$ with $\theta_i < \sigma_i$. Then $\sigma_2 \oplus \nu = \mu'$ for some ν.*

Proof. By Lemma 24 (*Rigidity of upper Cantor terms under simultaneous decreases*), μ' and $\rho \oplus \sigma_1 \oplus \sigma_2$ have the same terms in their Cantor normal forms at exponents at least e_2. Every term of σ_2 lies there, so subtracting its Cantor coefficients from those of μ' leaves an ordinal ν with $\sigma_2 \oplus \nu = \mu'$. □

`NatOrdinal.algebraicLE_right_of_forall_add_add_le`

Lemma 26 (Hessenberg-sum decomposition with equal last Cantor terms)**.** *Suppose $\sigma_1, \sigma_2 \neq 0$ have the same last Cantor term ω^e, with $e \neq 0$. If $\mu' < \rho \oplus \sigma_1 \oplus \sigma_2$ bounds every $\rho \oplus \theta_1 \oplus \theta_2$ with $\theta_i < \sigma_i$, then $\sigma_2 \oplus \nu = \mu'$ for some ν.*

Proof. Remove the common last term from both σ_i. The bound forces the terms in the Cantor normal form of μ' at exponents at least e to contain the remaining terms together with at least one copy of ω^e; strictness permits at most the two copies in the full sum. In either case these coefficients dominate those of σ_2, and coefficient subtraction gives ν with $\sigma_2 \oplus \nu = \mu'$. □

`NatOrdinal.algebraicLE_of_forall_add_add_le`

Lemma 27 (Upper Cantor-term bound for a Hessenberg sum)**.** *If $\rho \oplus \theta < \tau$ for every $\theta < \omega^\beta$, then the sum of the terms in the Cantor normal form of $\rho \oplus \omega^\beta$ whose exponents are at least β is at most τ.*

Proof. If this sum exceeded τ, monotonicity would force the sums of the terms in the Cantor normal forms of ρ and τ at exponents at least β to agree. Decomposing τ at β would then give $\tau \leq \rho \oplus \tau_{<\beta}$, where $\tau_{<\beta} < \omega^\beta$ is the sum of its remaining terms, contradicting the hypothesis. □

`NatOrdinal.partGE_add_wpow_le_of_forall_add_lt`

Lemma 28 (Intermediate ordinals below a Hessenberg sum)**.** *Let $\sigma \neq 0$ and suppose $\rho \oplus \theta < \tau$ for every $\theta < \sigma$. If $\tau < h' \leq \rho \oplus \sigma$, then there is $\rho' \leq \rho$ such that*

$$\rho' \oplus \sigma = h'.$$

Proof. Let ω^β be the last term of the Cantor normal form of σ. By Lemma 27 (*Upper Cantor-term bound for a Hessenberg sum*), after removing that term from σ, the sum of the terms of $\rho \oplus \sigma$ at exponents at least β is at most τ. Thus h' and $\rho \oplus \sigma$ have the same terms there. The terms of h' below β are bounded by those of ρ; combining them with the terms of ρ at exponents at least β gives $\rho' \leq \rho$ and $\rho' \oplus \sigma = h'$. □

`NatOrdinal.exists_le_add_eq_of_forall_add_lt`

2 Ordinal value and degree

The support of a generalised power series is well ordered, but its full order type is hard to carry through a product. The first step is to keep two simpler ordinal measurements. Berarducci's ordinal value records the final behaviour of the support near zero, once terms bounded away from zero no longer matter. The degree of a non-zero series is the first exponent in the Cantor normal form of its support order type; the degree of zero is $-\infty$. Their multiplication laws replace a product of series by Hessenberg's natural operations on ordinals.

Translated truncation gives a matching local operation: cut a series at one exponent, then shift that exponent back to zero. Modulo J, the convolution formula identifies a translated truncation of a product with a finite sum of products of translated truncations. For a principal series, every proper translated truncation has smaller ordinal value. More generally, if $v_J(b) \leq \omega^\beta$, then $v_J(b^{|\gamma}) < \omega^\beta$ for all $\gamma < 0$ sufficiently close to 0. These facts let the later induction compare local pieces by degree. The next chapter turns the Cantor degree of the ordinal value into a topological rank.

Lemma 29 (Convolution formula for translated truncations)**.** *Let K be a field. For $b, c \in K((\mathbb{R}))$ and $\gamma \in \mathbb{R}$,*

$$(bc)^{|\gamma} \equiv \sum_{\xi+\zeta=\gamma} b^{|\xi} c^{|\zeta} \bmod J.$$

Proof. For a cutoff γ, only finitely many pairs of points in the two closed supports can sum to γ. Below a sufficiently small neighbourhood of zero, every support pair contributing to the coefficient of bc is dominated by a unique such boundary pair. Regrouping the convolution product by that pair identifies the coefficients of $(bc)^{|\gamma}$ with the finite sum of $b^{|\xi} c^{|\zeta}$; equality near zero is precisely congruence modulo J. This is the proof of [Ber00, Lemma 7.5(2)]; its argument does not use $\gamma \leq 0$, so it gives the displayed formula at every real cutoff. ☐

`Berarducci.germAt_mul`

Fact 30 (Principal representatives of P_α (LM24, Remark 7.2.4))**.** *Every nonzero element of P_α is represented by a principal series p of exact degree α.*

Proof. A nonzero class in P_α has a representative b with $v_J(b) = \omega^\alpha$. The principal-part theorem replaces b by a principal series p of degree α with $v_J(b - p) < \omega^\alpha$; this is exactly equality in the quotient. ☐

`Berarducci.exists_principal_representative_of_ne_zero`

Fact 31 (Support-tail characterization of the ordinal value (Ber00, Definition 5.2))**.** *Let K be a field and let $b \in K((\mathbb{R}^{\leq 0}))$. If $1 < v_J(b)$, then there is $\eta < 0$ such that*

$$\mathrm{ot}(\mathrm{supp}(b) \cap (\eta, 0)) = v_J(b).$$

Proof. By the definition of v_J, choose $c \equiv b \pmod{J + K}$ whose support has order type $v_J(b)$. The negative coefficients of b and c agree on some interval $(\eta, 0)$, so their support tails there coincide. Minimality of $v_J(b)$ bounds the order type of this tail from below, while its inclusion in $\mathrm{supp}(c)$ bounds it from above. Hence both bounds are equalities. ☐

`Berarducci.exists_negativeSupportTail_orderType_eq_ordinalValue`

Fact 32 (Multiplicative property of the ordinal value (Ber00, Theorem 9.7))**.** *Let K be a field of characteristic zero. For all $b, c \in K((\mathbb{R}^{\leq 0}))$,*

$$v_J(bc) = v_J(b) \odot v_J(c).$$

Proof. The cases $v_J(b) = 0$, $v_J(c) = 0$, $v_J(b) = 1$, and $v_J(c) = 1$ follow from the ideal J and multiplication by a series of ordinal value one. In the remaining case, Fact 31 (*Support-tail characterization of the ordinal value (Ber00, Definition 5.2)*) bounds every sufficiently high proper translated truncation below the value of its factor. Together with Lemma 29 (*Convolution formula for translated truncations*), this supplies Berarducci's well-founded induction on finite products of factors of ordinal value greater than one. Applying the resulting identity to b, c gives the formula. ☐

`Berarducci.ordinalValue_mul`

Fact 33 (Multiplicativity of order type for weakly principal series (Ber00, Corollary 9.9; LM24, Fact 3.4.1))**.** *Let K be a field of characteristic zero. If $b, c \in K((\mathbb{R}^{\leq 0}))$ are weakly principal, then*

$$\mathrm{ot}(bc) = \mathrm{ot}(b) \odot \mathrm{ot}(c).$$

Proof. Translate each factor so that the supremum of its support is zero. Translation preserves the three support order types and places both factors outside J. For a weakly principal series outside J, its support order type equals its ordinal value. Fact 32 (*Multiplicative property of the ordinal value (Ber00, Theorem 9.7)*) gives the lower bound for the product, while containment of its support in the sum of the two supports gives the reverse bound. □

`HahnSeries.Nonpositive.orderTypeMultiplicativeOnWeaklyPrincipal`

Fact 34 (Multiplicativity of the degree (LM24, Theorem D))**.** *For $b, c \in K((\mathbb{R}^{\leq 0}))$,* $\deg(bc) = \deg(b) \oplus \deg(c)$.

Proof. Decompose each nonzero series into a principal leading summand and a remainder of no larger degree. By Fact 33 (*Multiplicativity of order type for weakly principal series (Ber00, Corollary 9.9; LM24, Fact 3.4.1)*), the support order types of the two principal summands multiply. Together with the strict degree bounds for the remaining products, this shows that the leading term of bc has degree $\deg(b) \oplus \deg(c)$. Hence $\deg(bc) = \deg(b) \oplus \deg(c)$. The zero cases follow from the ring laws. □

`HahnSeries.Nonpositive.degree_mul`

Lemma 35 (Decrease of the ordinal value under translated truncation)**.** *Let $\beta < \omega_1$ and $b \in K((\mathbb{R}^{\leq 0}))$. If $v_J(b) \leq \omega^\beta$, then*

$$v_J(b^{|\gamma}) < \omega^\beta$$

for all $\gamma < 0$ sufficiently close to 0.

Proof. If $b \in J + K$, all sufficiently late translated truncations lie in J. Otherwise $1 < v_J(b)$, and Fact 31 (*Support-tail characterization of the ordinal value (Ber00, Definition 5.2)*), followed by shortening the interval, gives a tail $(\eta, 0)$ of order type $v_J(b)$. For a later cutoff γ, either the support has a gap immediately below γ, giving value at most $1 < \omega^\beta$, or $\mathrm{supp}(b) \cap (\eta, \gamma)$ is both a final segment of the support below γ and a proper initial segment of the tail $(\eta, 0)$. Hence

$$v_J(b^{|\gamma}) \leq \mathrm{ot}(\mathrm{supp}(b) \cap (\eta, \gamma)) < v_J(b) \leq \omega^\beta.$$

□

`Berarducci.eventually_ordinalValue_translatedTruncation_lt_wpow_of_ordinalValue_le`

3 Cantor–Bendixson ranks of supports

The ordinal measurements tell us how supports behave under multiplication, but a polynomial relation calls for more local information. For series with real exponents, that information is visible in the closed support near zero. A Cantor–Bendixson derivative removes the isolated points of a set. A point belonging to the closed set has rank r precisely when it survives the r-th derivative and disappears at the $(r + 1)$-st derivative.

This chapter proves that the value defined from this closed support is multiplicative, then identifies it with Berarducci's ordinal value. When zero belongs to the closed support, its Cantor–Bendixson rank agrees with the Cantor degree of the ordinal value; when zero lies outside, that degree is $-\infty$. This description is useful because the derivatives can pick out the points of an exact rank. The next chapter uses them to inspect the leading degree of a homogeneous polynomial relation.

Definition 36 (The Cantor–Bendixson value at exponent zero)**.** Let G be a linearly ordered set with zero and its order topology, let R be a set with zero, and let $b \in R((G))$ be a generalised power series. Write $C = \mathrm{cl}(\mathrm{supp}(b))$. If $\mathrm{rk}_C(0)$ denotes the Cantor–Bendixson rank of 0 in C, define

$$V_{\mathrm{CB}}(b) = \begin{cases} \omega^{\mathrm{rk}_C(0)}, & 0 \in C, \\ 0, & 0 \notin C. \end{cases}$$

Proof. The closure of a well-ordered support is again well ordered, so its Cantor–Bendixson point rank is defined. The displayed alternatives are the two branches of the definition. ☐

`HahnSeries.cantorBendixsonValue`

Lemma 37 (Finite convolution of translated truncations)**.** *Let R be a ring, let G be a nontrivial ordered abelian group equipped with a compatible additive uniformity and its order topology, and assume that G is Cauchy complete. Let $b, d \in R((G))$. For $\gamma \in G$, let F_γ be the finite addition fiber*

$$F_\gamma = \{(x, y) \in \mathrm{cl}(\mathrm{supp}(b)) \times \mathrm{cl}(\mathrm{supp}(d)) : x + y = \gamma\}.$$

Then

$$V_{\mathrm{CB}}((bd)^{|\gamma}) = V_{\mathrm{CB}}\left(\sum_{(x,y)\in F_\gamma} b^{|x} d^{|y}\right).$$

Proof. Closed well-ordered supports have finite addition fibers. In a neighbourhood of γ, each coefficient of bd is the finite sum of the products of the corresponding truncations indexed by F_γ. After translating γ to 0, the difference between the two displayed series vanishes on a neighbourhood of 0. By Definition 36 (*The Cantor–Bendixson value at exponent zero*), its value is zero, and adding it does not change the Cantor–Bendixson rank at 0. The two values are therefore equal. ☐

`HahnSeries.cantorBendixsonValue_convolution`

Lemma 38 (Reconstruction from translated truncations of fixed rank)**.** *Let R be an additive monoid, let G be a nontrivial ordered topological abelian group with its order topology, and let $b, d \in R((G))$. Suppose* $\mathrm{supp}(b) \subseteq G^{\le 0}$*. Let a, c, r be ordinals with $r > 0$ and*

$$V_{\mathrm{CB}}(b) = \omega^{a+r}.$$

Suppose that, for every $\gamma < 0$ sufficiently close to 0,

$$V_{\mathrm{CB}}(b^{|\gamma}) = \omega^a \implies V_{\mathrm{CB}}(d^{|\gamma}) \ge \omega^c.$$

Then

$$V_{\mathrm{CB}}(d) \ge \omega^{c+r}.$$

Proof. By Definition 36 (*The Cantor–Bendixson value at exponent zero*), the hypothesis on b says that 0 has Cantor–Bendixson rank $a+r$ in its closed support. At every nearby point of exact rank a, the translated-truncation hypothesis places that point in the c-th derivative of the closed support of d. Cantor–Bendixson reconstruction therefore places 0 in its $(c+r)$-th derivative, which is the stated value bound. □

`HahnSeries.cantorBendixsonValue_reconstruction`

Lemma 39 (Cantor–Bendixson derivatives of sums of well-ordered sets)**.** *Let S, T be closed well-ordered subsets of a nontrivial ordered abelian group G, equipped with a compatible additive uniformity and its order topology, and assume that G is Cauchy complete. For every ordinal α,*

$$(S+T)^{(\alpha)} \subseteq \left\{ z : \begin{array}{l} z = x + y \text{ for some } x \in S, y \in T, \\ \alpha \le \mathrm{rk}_S(x) \oplus \mathrm{rk}_T(y) \end{array} \right\}.$$

Proof. On $S \times T$, the natural sum of the two point ranks strictly decreases in a punctured neighbourhood of each point. Transfinite Cantor–Bendixson induction therefore bounds the derivative rank on the product. Addition $S \times T \to S + T$ is closed and has finite fibers; lifting derivatives through this map gives the stated summands and rank bound. □

`TopologicalSpace.Closeds.cantorBendixson_add_subset`

Lemma 40 (Product upper bound for the Cantor–Bendixson value)**.** *Let R be a semiring, not necessarily unital or associative, and let G be a nontrivial ordered abelian group equipped with a compatible additive uniformity and its order topology. Assume that G is Cauchy complete. If $b, d \in R((G^{\le 0}))$, then*

$$V_{\mathrm{CB}}(bd) \le V_{\mathrm{CB}}(b) \odot V_{\mathrm{CB}}(d),$$

where $\odot$ is Hessenberg's natural product.

Proof. By Definition 36 (*The Cantor–Bendixson value at exponent zero*), the assertion is trivial if 0 is outside the closed support of bd; otherwise its value is ω to the rank there. The closed support of bd is contained in the sum of the closed supports of b and d. By Lemma 39 (*Cantor–Bendixson derivatives of sums of well-ordered sets*), a point of rank α in this sum lifts to x and y whose ranks have natural sum at least α. Nonpositivity and $x+y=0$ force $x=y=0$. Exponentiating the resulting rank inequality by ω turns natural sum into natural product and gives the bound. □

`HahnSeries.cantorBendixsonValue_mul_le`

Lemma 41 (Pure-power cancellation for the Cantor–Bendixson value)**.** *Let R be a characteristic-zero domain, let G be a nontrivial complete ordered abelian group equipped with a compatible additive uniformity and its order topology, and let $b \in R((G^{\le 0}))$. Suppose $V_{\mathrm{CB}}(b) = B > 1$, where B is additively principal, and write $B = \rho_B \odot \pi_B$ for its residual factor and its final infinite multiplicatively principal factor. Let $m \in \mathbb{N}$.*

Suppose that, for every $\gamma < 0$ sufficiently close to 0,

$$V_{\mathrm{CB}}(b^{|\gamma}) = \rho_B \quad \Longrightarrow \quad V_{\mathrm{CB}}(b^{|\gamma} b^m) = B^{\odot m} \odot \rho_B.$$

Then

$$V_{\mathrm{CB}}(b^{m+1}) = B^{\odot(m+1)}.$$

Here products and powers marked by $\odot$ are Hessenberg's natural operations.

Proof. The translated truncation of b^{m+1} is the main term $(m+1)b^{|\gamma}b^m$ plus a remainder of smaller Cantor–Bendixson value. Characteristic zero preserves the value of the nonzero coefficient $m+1$, so the local hypothesis computes the value of the main term and hence of the whole truncation. Applying Lemma 38 (*Reconstruction from translated truncations of fixed rank*) supplies the required lower bound at 0; the reverse inequality follows by iterating Lemma 40 (*Product upper bound for the Cantor–Bendixson value*) over the power. ☐

`HahnSeries.cantorBendixsonValue_pow_eq_of_eventually`

Lemma 42 (Power-times-factor cancellation for the Cantor–Bendixson value)**.** *Under the coefficient and exponent-group hypotheses of Lemma 41 (*Pure-power cancellation for the Cantor–Bendixson value*), let $b, c \in R((G^{\leq 0}))$ have additively principal values $V_{\mathrm{CB}}(b) = B > 1$ and $V_{\mathrm{CB}}(c) = C > 1$. Write $B = \rho_B \odot \pi_B$ and $C = \rho_C \odot \pi_C$ as above, and suppose $\pi_B \leq \pi_C$. If, for every $\gamma < 0$ sufficiently close to 0,*

$$V_{\mathrm{CB}}(b^{|\gamma}) = \rho_B \quad \Longrightarrow \quad V_{\mathrm{CB}}\Big(b^{|\gamma}b^m c^2\Big) = B^{\odot m} \odot \rho_B \odot C \odot C,$$

then

$$V_{\mathrm{CB}}(b^{m+1}c) = B^{\odot(m+1)} \odot C.$$

Proof. Put $d = b^{m+1}c$. Multiply the translated-truncation expansion of d by c. Its main term is $(m+1)b^{|\gamma}b^m c^2$; the two remaining terms have strictly smaller value by Lemma 40 (*Product upper bound for the Cantor–Bendixson value*), the ordering $\pi_B \leq \pi_C$, and the ordinal factorisation of B and C. Thus the hypothesis computes $V_{\mathrm{CB}}(cd^{|\gamma})$. If $V_{\mathrm{CB}}(d^{|\gamma})$ were too small, the same upper bound would contradict this computation. The resulting local lower bound is lifted to 0 by Lemma 38 (*Reconstruction from translated truncations of fixed rank*); the product upper bound gives the reverse inequality. ☐

`HahnSeries.cantorBendixsonValue_pow_mul_eq_of_eventually`

Theorem 43 (Multiplicativity of the Cantor–Bendixson value)**.** *Let R be a characteristic-zero domain and let G be a nontrivial ordered abelian group equipped with a compatible additive uniformity and its order topology. Assume that G is Cauchy complete. For all $b, c \in R((G^{\leq 0}))$,*

$$V_{\mathrm{CB}}(bc) = V_{\mathrm{CB}}(b) \odot V_{\mathrm{CB}}(c),$$

where $\odot$ is Hessenberg's natural product.

Proof. If either value is 0, the product upper bound forces the product value to be 0. A factor of value 1 is a nonzero scalar at exponent 0 plus a remainder of value 0, so it preserves the other value. For values greater than 1, argue by well-founded induction on the finite multiset of factors. Choose a factor for which the final multiplicatively principal factor is least, and among ties choose one of greatest value. At a cutoff where its translated truncation has the residual value, that truncation has smaller value, so the induction hypothesis computes every required local product. If no other factors remain, apply Lemma 41 (*Pure-power cancellation for the Cantor–Bendixson value*); otherwise combine the remaining factors and apply Lemma 42 (*Power-times-factor cancellation for the Cantor–Bendixson value*). In both cases the result is the natural product of the factor values, and the two-factor statement follows. ☐

`HahnSeries.cantorBendixsonValue_mul`

Theorem 44 (Multiplicativity of the Cantor–Bendixson degree)**.** *Let R be a characteristic-zero domain and let G be a nontrivial ordered abelian group equipped with a compatible additive uniformity and its order topology. Assume that G is Cauchy complete. For all $b, c \in R((G^{\le 0}))$,*

$$\deg\bigl(V_{\mathrm{CB}}(bc)\bigr) = \deg\bigl(V_{\mathrm{CB}}(b)\bigr) \oplus \deg\bigl(V_{\mathrm{CB}}(c)\bigr),$$

where deg *is Cantor degree, with value $-\infty$ at 0, and $\oplus$ is Hessenberg's natural sum.*

Proof. Take Cantor degree in Theorem 43 (*Multiplicativity of the Cantor–Bendixson value*). Cantor degree sends Hessenberg's natural product to Hessenberg's natural sum. □

`HahnSeries.Nonpositive.cantorBendixsonDegreeValuation_mul`

Lemma 45 (Cantor–Bendixson rank at a strict supremum)**.** *Let $S \subseteq \mathbb{R}$ be well ordered and let z be its least upper bound, with $z \notin S$. If $\alpha \neq 0$ and $\mathrm{ot}(S) = \omega^\alpha$, then*

$$\mathrm{rk}_{\mathrm{CB},\mathrm{cl}(S)}(z) = \alpha.$$

Proof. The closure of S is again well ordered. Its increasing enumeration is a homeomorphism from an ordinal interval, and the strict initial segment below z still has order type ω^α. In an ordinal interval, the α-th Cantor–Bendixson derivative contains the point indexed by ω^α, while the $(\alpha+1)$-st derivative does not. Transport these two statements through the homeomorphism to the closure of S. □

`Set.IsPWO.cantorBendixsonRank_closure_eq_of_orderType_eq_opow`

Lemma 46 (Cantor–Bendixson formula for the ordinal value)**.** *Let $b \in K((\mathbb{R}^{\le 0}))$. If zero does not belong to the closed support of b, then $v_J(b) = 0$; otherwise*

$$v_J(b) = \omega^{\mathrm{rk}_{\mathrm{CB},\mathrm{cl}(\mathrm{supp}(b))}(0)}.$$

Proof. By Definition 36 (*The Cantor–Bendixson value at exponent zero*), the right-hand side is zero off the closed support at 0 and otherwise records its Cantor–Bendixson rank. For $v_J(b) = 0$ or 1 the assertion is the definition of J and of the congruence class modulo $J+K$. If $v_J(b) = \omega^\alpha > 1$, then Fact 31 (*Support-tail characterization of the ordinal value (Ber00, Definition 5.2)*) gives $\eta < 0$ for which $\mathrm{supp}(b) \cap (\eta, 0)$ has order type ω^α. Zero is its strict supremum, so Lemma 45 (*Cantor–Bendixson rank at a strict supremum*) gives rank α at zero in the closure of this tail. The closed support of b agrees with that closure on a neighbourhood of zero. Cantor–Bendixson rank is local with respect to closed sets, so it has the same value in the closed support. □

`Berarducci.ordinalValue_eq_cantorBendixsonValue`

Lemma 47 (Cantor–Bendixson formula for $\deg_J$)**.** *Let $b \in K((\mathbb{R}^{\le 0}))$, and let $\deg_J(b)$ be the Cantor degree of $v_J(b)$, with value $-\infty$ when $v_J(b) = 0$. Then*

$$\deg_J(b) = \begin{cases} -\infty, & 0 \notin \mathrm{cl}(\mathrm{supp}(b)), \\ \mathrm{rk}_{\mathrm{CB},\mathrm{cl}(\mathrm{supp}(b))}(0), & 0 \in \mathrm{cl}(\mathrm{supp}(b)). \end{cases}$$

Proof. By Theorem 43 (*Multiplicativity of the Cantor–Bendixson value*), taking Cantor degree of V_{CB} gives the multiplicative degree on the right. Apply Cantor degree to Lemma 46 (*Cantor–Bendixson formula for the ordinal value*). The Cantor degree of 0 is $-\infty$, and the Cantor degree of ω^α is α. □

`Berarducci.ordinalValueDegree_eq_cantorBendixsonDegree`

4 Algebraic independence in graded rings

The last chapter turned degree into a topological rank that can be checked locally. Now suppose that a minimal homogeneous generating family satisfied a polynomial relation. Choose the first degree where evaluation is not injective, and apply Cantor–Bendixson derivatives to the leading supports of representatives. At a successor degree, the Leibniz rule turns this into identities between the ordinary partial derivatives of the polynomial. A limit degree needs separate control of the variables with maximal weight.

Homogeneous ideals, a local-to-global argument, and polynomial syzygies combine these local identities into a contradiction. The result is a general test for algebraic independence of a minimal homogeneous generating family. The test asks for precise facts about the translated truncations of chosen representatives. Those facts are not automatic for generalised power series, so the next two chapters prove them.

Lemma 48 (Discrete finite unions of Cantor–Bendixson rank sets)**.** *Let G be a nontrivial ordered abelian group with compatible additive uniformity and order topology, and let R be a characteristic-zero domain. Assume that G is Cauchy complete. Write ν for the Cantor–Bendixson degree on $R((G^{\leq 0}))$. For a series q and an ordinal ρ, let*

$$L_\rho(q) = \{\gamma \in \overline{\operatorname{supp}(q)} : \operatorname{rk}_{\overline{\operatorname{supp}(q)}}(\gamma) = \rho\}.$$

If B is finite and $\nu(q_b) \leq \rho_b + 1$ for every $b \in B$, then there is $\eta < 0$ such that

$$\bigcup_{b \in B} \big(L_{\rho_b}(q_b) \cap (\eta, 0)\big)$$

is discrete.

Proof. By Theorem 43 (*Multiplicativity of the Cantor–Bendixson value*), the Cantor–Bendixson degree is the multiplicative degree denoted by ν. Each exact-rank set is discrete. Moreover, once the translated truncations of q_b have degree at most ρ_b, the exact-rank set agrees locally with its closure. Choose such a negative cutoff for each b and take their maximum. Above this common cutoff, each set has a neighbourhood disjoint from every other set at any point it does not contain. The finite union is therefore discrete. ☐

`HahnSeries.Nonpositive.exists_isDiscrete_iUnion_rankLevelSet`

Lemma 49 (Cofactors from local data by well-founded induction)**.** *Let K be a field of characteristic zero and let G be a complete densely ordered abelian group with a decreasing well-founded neighbourhood basis (U_i) of open convex additive subgroups. Let b_i represent homogeneous classes x_i of degrees w_i in the Cantor–Bendixson associated graded ring, with*

$$\nu(b_i) \leq w_i, \qquad \nu(b_i^{|y}) < w_i \quad (y < 0).$$

Assume homogeneous evaluation at (x_i) is generating and injective below α.

Let Q_j be finitely many weighted-homogeneous polynomials of degrees σ_j. Fix $\tau < \mu < \alpha$ and ordinals $P_j(\beta)$ satisfying

$$P_j(\beta) \oplus \sigma_j = \beta \quad (\tau < \beta \leq \mu), \qquad P_j(\mu) \oplus \theta < \tau \quad (\theta < \sigma_j).$$

Suppose

$$\nu(u) \leq \mu, \qquad \nu(u^{|y}) < \mu \quad (y < 0),$$

and, at every negative cutoff, $u^{|y}$ *is congruent modulo series bounded away from zero to an evaluated polynomial all of whose monomials have weight below* α *and whose terms of degree at least* τ *lie in the ideal generated by the* Q_j*. Then there are series* c_j *such that*

$$\nu(c_j) \le P_j(\mu), \qquad \nu\left(u - \sum_j c_j Q_j(b)\right) \le \tau \oplus 1.$$

Proof. By Theorem 43 (*Multiplicativity of the Cantor–Bendixson value*), the Cantor–Bendixson degree defines the associated graded ring used below. Partition the closed support of u inside the negative half-line into ordered disjoint convex pieces whose ranks are strictly smaller than the rank of the piece centre. On each piece Lemma 3 (*Homogeneous decomposition in a finitely generated graded ideal*) first decomposes the homogeneous ideal relation into cofactors of complementary degrees; the well-founded induction then produces local series cofactors with the prescribed degree bounds. Translate them back to their piece centres. Separation of the convex carriers makes the translated families summable, so their Hahn sums c_j preserve the bounds $P_j(\mu)$. The local residual bounds combine on the same carriers, giving degree at most $\tau \oplus 1$ for the global residual. □

`HahnSeries.Nonpositive.exists_cofactors_degree_le_add_one_of_properly_locallyIdeal`

Lemma 50 (Local ideal membership in the associated graded ring)**.** *Under all the hypotheses of Lemma 49 (*Cofactors from local data by well-founded induction*), suppose in addition that* $\tau \oplus 1 < \mu$*. If* u *represents a homogeneous class* $\bar{u}$ *of degree* μ*, then*

$$\bar{u} \in \big(Q_j(x) : j\big)$$

in the Cantor–Bendixson associated graded ring.

Proof. By Lemma 49 (*Cofactors from local data by well-founded induction*), choose cofactors c_j of degree at most $P_j(\mu)$ whose residual has degree at most $\tau \oplus 1 < \mu$. The degree-μ class of the residual is therefore zero. Multiplication of represented homogeneous classes and $P_j(\mu) \oplus \sigma_j = \mu$ identify the remaining degree-μ class with $\sum_j [c_j] Q_j(x)$. □

`HahnSeries.Nonpositive.homogeneousClass_mem_span_of_properly_locallyIdeal`

Lemma 51 (Polynomial syzygies from homogeneous ideal membership)**.** *Let* x_i *be homogeneous elements of a graded* K*-algebra. Assume homogeneous polynomial evaluation at* (x_i) *is generating and injective below* α*. Let* A *and the finitely many* Q_j *be weighted-homogeneous of degrees* $m < \alpha$ *and* σ_j*. If*

$$A(x) \in \big(Q_j(x) : j\big),$$

then there are polynomials C_j *such that*

$$A = \sum_j C_j Q_j,$$

and C_j *is weighted-homogeneous of degree* β *whenever* $\beta \oplus \sigma_j = m$*.*

Proof. By Theorem 43 (*Multiplicativity of the Cantor–Bendixson value*), the Cantor–Bendixson degree defines the associated graded ring in which the ideal is formed. By Lemma 3 (*Homogeneous decomposition in a finitely generated graded ideal*), homogeneous ideal membership writes $A(x)$ as a sum of the $Q_j(x)$ with homogeneous cofactors of complementary degrees. Homogeneous generation below α represents each cofactor by a polynomial of the same degree. The resulting polynomial combination and A have the same evaluation and are homogeneous of degree m; injectivity in that degree makes them equal. □

`HahnSeries.Nonpositive.exists_eq_sum_mul_of_class_mem_span`

Lemma 52 (Local-to-global principle for homogeneous ideal membership)**.** *Let K be a field of characteristic zero and let G be a complete densely ordered abelian group with a decreasing well-founded neighbourhood basis (U_i) of open convex additive subgroups. Let b_i represent homogeneous classes x_i of degrees w_i in the Cantor–Bendixson associated graded ring, with*

$$\nu(b_i) \leq w_i, \qquad \nu(b_i^{|y}) < w_i \quad (y < 0).$$

Assume homogeneous evaluation at (x_i) is generating and injective below α.

Let Θ and finitely many Q_j be weighted homogeneous polynomials of degrees μ and $\sigma_j \neq 0$, where $\mu < \alpha$. Suppose $\rho_j \oplus \sigma_j = \mu$, $\tau \oplus 1 < \mu$, and the upper Cantor terms of μ at and above the exponent of the last Cantor term of σ_j have sum at most τ. Assume

$$\nu(\Theta(b)) \leq \mu, \qquad \nu(\Theta(b)^{|\gamma}) < \mu \quad (\gamma < 0).$$

If, for every negative cutoff sufficiently close to zero, there is a polynomial all of whose monomials have weight below α, congruent to $\Theta(b)^{|\gamma}$ modulo series bounded away from zero, and whose terms of degree at least τ lie in the ideal generated by the Q_j, then

$$\Theta = \sum_j C_j Q_j,$$

where C_j is weighted homogeneous of every degree β satisfying $\beta \oplus \sigma_j = \mu$; in particular it has degree ρ_j.

Proof. Choose a cutoff after which the local ideal condition holds and replace $\Theta(b)$ by its strict tail there. This preserves the represented degree-μ class, the bound $\nu(\Theta(b)) \leq \mu$, and the strict translated-truncation bounds, while translated truncations below the cutoff vanish. Thus the local condition holds at every negative cutoff. Lemma 17 (*Separation below the last Cantor term*) turns each window bound into the separation inequality below the last Cantor term, and Lemma 28 (*Intermediate ordinals below a Hessenberg sum*) identifies the possible complementary degrees ρ_j. Lemma 50 (*Local ideal membership in the associated graded ring*) combines the local representatives and places the resulting homogeneous class in the ideal generated by the Q_j. Then Lemma 51 (*Polynomial syzygies from homogeneous ideal membership*) lifts that graded ideal membership to the stated polynomial identity. □

`HahnSeries.Nonpositive.exists_eq_sum_mul_of_eventually_componentsGE_mem_of_windows`

Lemma 53 (The successor formula for the Cantor–Bendixson derivation)**.** *Let ν be the Cantor–Bendixson degree on $R((G^{\leq 0}))$, and let ∂_{CB} be the additive map from gr_ν to functions $G \to \mathrm{gr}_\nu$ modulo equality on a left neighbourhood of 0. If $\nu(b) \leq \alpha + 1$, then*

$$\partial_{\mathrm{CB}}\big(\mathrm{in}_{\alpha+1}(b)\big) = \Big[\gamma \longmapsto \mathrm{in}_\alpha\big(b^{|\gamma}\big)\Big]_{\gamma \to 0^-},$$

where $b^{|\gamma}$ is the translated truncation of b at γ.

Proof. By Theorem 43 (*Multiplicativity of the Cantor–Bendixson value*), the Cantor–Bendixson degree defines the multiplicative filtration and associated graded ring used here. On the component of degree $\alpha + 1$, the map ∂_{CB} is defined by taking the degree-α class of each translated truncation. Including those classes in the associated graded ring gives the displayed equality. $\square$

`HahnSeries.Nonpositive.cantorBendixsonGradedDerivation_homogeneousMk_succ`

Lemma 54 (The Leibniz rule for the Cantor–Bendixson derivation)**.** *For all $x, y \in \mathrm{gr}_\nu$,*

$$\partial_{\mathrm{CB}}(xy) = \partial_{\mathrm{CB}}(x)y + x\partial_{\mathrm{CB}}(y),$$

where elements of gr_ν on the right are regarded as constant functions near 0.

Proof. Decompose x and y into homogeneous components. On successor components, Lemma 53 (*The successor formula for the Cantor–Bendixson derivation*) turns the identity into the translated-truncation product formula. The derivative vanishes on components whose constant Cantor coefficient is zero. Additivity then gives the formula for arbitrary x and y. $\square$

`HahnSeries.Nonpositive.cantorBendixsonGradedDerivation_mul`

Lemma 55 (Prescribing the Cantor–Bendixson derivative on one exact-rank set)**.** *Let $q \in \mathrm{gr}_\nu$ be homogeneous of degree c, and let a series p have Cantor–Bendixson degree at most $\delta + 1$. Prescribe at every point of exact rank δ in* $\mathrm{supp}(p)$ *a homogeneous class a_γ whose degree β satisfies $\beta + c = \delta$, taking $a_\gamma = 0$ when no such β exists. Then there is $z \in \mathrm{gr}_\nu$ such that qz is homogeneous of degree $\delta + 1$ and the Cantor–Bendixson derivative of z agrees near 0 with a_γ on that exact-rank set and is zero away from it.*

Proof. If no β satisfies $\beta + c = \delta$, take $z = 0$. Otherwise, all prescribed values have the same degree β. Choose component representatives and place them on pairwise separated left intervals ending at the exact-rank points of p. Their Hahn sum has degree at most $\beta + 1$ and has the prescribed translated-truncation classes. Its degree-$(\beta + 1)$ class is z; the graded product has degree $c + (\beta + 1) = \delta + 1$, and Lemma 53 (*The successor formula for the Cantor–Bendixson derivation*) gives the required near-zero equality. $\square$

`HahnSeries.Nonpositive.exists_grading_mul_and_derivation_eq_rankLevel`

Lemma 56 (The derivative criterion for successor ideal membership)**.** *Let (q_j) be a finite homogeneous family in gr_ν, of degrees c_j with zero constant Cantor coefficient. Let x be homogeneous of degree $\delta + 1$. If the Cantor–Bendixson derivative of x is represented near 0 by a function f satisfying*

$$f(\gamma) \in (q_j : j) \qquad \text{for every } \gamma \in G,$$

then $x \in (q_j : j)$.

Proof. Decompose each $f(\gamma)$ homogeneously in the generators q_j using Lemma 3 (*Homogeneous decomposition in a finitely generated graded ideal*). On the exact-rank set of a representative of x, apply Lemma 55 (*Prescribing the Cantor–Bendixson derivative on one exact-rank set*) to each coefficient. The resulting classes z_j make $y = \sum_j q_j z_j$ homogeneous of degree $\delta + 1$ and give x and y the same Cantor–Bendixson derivative near 0. The derivative is injective in successor degree, so $x = y \in (q_j : j)$. Products are differentiated using Lemma 54 (*The Leibniz rule for the Cantor–Bendixson derivation*). $\square$

Lemma 57 (Prescribing the Cantor–Bendixson derivative on a discrete set)**.** *Let K be a field of characteristic zero and G a nontrivial complete ordered abelian group with compatible additive uniformity and order topology. Write ν for the Cantor–Bendixson degree on $K((G^{\leq 0}))$. Suppose $S \subseteq G^{\leq 0}$ is partially well ordered and discrete, and that $\overline{S} \subseteq S$ on some left neighbourhood of 0. If every $a(\gamma)$ is homogeneous of degree ρ in gr_ν, then there is a homogeneous $s \in \mathrm{gr}_\nu$ of degree $\rho + 1$ whose Cantor–Bendixson derivative agrees near 0 with*

$$\gamma \longmapsto \begin{cases} a(\gamma), & \gamma \in S, \\ 0, & \gamma \notin S. \end{cases}$$

Proof. Regard each prescribed value as an element of the degree-ρ component. Discreteness supplies pairwise separated left intervals about the points of S. Place a representative of $a(\gamma)$ in the interval ending at γ and sum the resulting Hahn series. Partial well-ordering makes the family summable. Because the closure of S adds no points near 0, translated truncation at a point of S recovers the prescribed class and gives zero elsewhere. The resulting series has degree at most $\rho+1$, and its degree-$(\rho+1)$ class is the required s by Lemma 53 (*The successor formula for the Cantor–Bendixson derivation*). ☐

HahnSeries.Nonpositive.exists_derivation_eq_of_isDiscrete

Lemma 58 (Leading-coefficient obstruction for homogeneous relations)**.** *Let K be a field of characteristic zero and G a nontrivial complete ordered abelian group with compatible additive uniformity and order topology. Let x_i be a minimal homogeneous generating system for the associated graded ring of the degree filtration, of weights w_i. Choose series b_i representing x_i such that*

$$\deg(b_i) \leq w_i, \qquad \deg(b_i^{|y}) < w_i \quad \textit{for every } y < 0.$$

Assume evaluation at x_i is injective on every homogeneous degree below α.

Let $F \in K[X_i : i \in I]$ be weighted homogeneous of degree α with $F(x) = 0$. Choose a variable X_{B_0} occurring in F, of maximal weight among its variables, and suppose $w_{B_0} < \alpha$. Write

$$F = \sum_{k=0}^{D} H_k X_{B_0}^k, \qquad \Delta + D w_{B_0} = \alpha.$$

If $\Delta = 0$, or if the last Cantor term of Δ is at most the last Cantor term of w_{B_0}, then these hypotheses are inconsistent.

Proof. By Theorem 43 (*Multiplicativity of the Cantor–Bendixson value*), the Cantor–Bendixson degree defines the associated graded ring in which the weighted-homogeneous relation is evaluated.

Suppose first that $\Delta \neq 0$. Then $H_D \neq 0$ is homogeneous of degree $\Delta < \alpha$. Injectivity below α shows that $H_D(x) \neq 0$, so $H_D(b)$ has degree exactly Δ. By Lemma 4 (*Weighted-homogeneous polynomial representatives*), each proper translated truncation has a representing polynomial. Expanding F in X_{B_0} and using the finite convolution formula identifies the coefficient of $X_{B_0}^D$ in that polynomial with the polynomial representing the corresponding translated truncation of $H_D(b)$.

Since $F(x) = 0$, the translated truncations of $F(b)$ have their representing polynomials bounded in weighted degree by some $q < \alpha$. The hypothesis on the last Cantor terms and

Lemma 21 (*Cofinality below a Hessenberg sum*) give $\rho < \Delta$ with $q \leq \rho \oplus Dw_{B_0}$. A sufficiently late translated truncation of $H_D(b)$ has degree ρ, so its representing polynomial has weighted degree ρ. Its contribution in degree $\rho \oplus Dw_{B_0}$ contradicts the bound q.

If $\Delta = 0$, then H_D is a nonzero scalar and $D \geq 2$. The polynomial $H_{D-1} + DH_D X_{B_0}$ is nonzero and homogeneous of degree w_{B_0}. Injectivity below α makes its value at b have that exact degree. The coefficient of $X_{B_0}^{D-1}$ in the translated-truncation polynomial is the polynomial representing its translated truncation, and the same argument gives the contradiction. □

`HahnSeries.Germ.false_of_aeval_eq_zero_of_leastTerm_le`

Lemma 59 (Injectivity of evaluation when the degree is a limit ordinal). *Let K be a field of characteristic zero and G a nontrivial complete ordered abelian group equipped with a compatible additive uniformity whose topology is the order topology. Let x_i be a minimal homogeneous generating system for the associated graded ring of the degree filtration, with weights w_i, and choose series b_i representing x_i in degree w_i. Let $\alpha \neq 0$ have zero constant Cantor coefficient.*

Assume first that the following leading-coefficient case is impossible. If $0 \neq F \in K[X_i : i \in I]$ is weighted homogeneous of degree α, $F(x) = 0$, X_{B_0} has maximal weight among the variables of F, and

$$\Delta + Dw_{B_0} = \alpha, \qquad D = \deg_{B_0} F,$$

where $w_{B_0} < \alpha$, then $\Delta = 0$ or the last Cantor term of Δ being at most the last Cantor term of w_{B_0} gives a contradiction.

In the complementary case, suppose the following two conclusions hold whenever ordinals $\beta, \lambda_0, \alpha_1$ and data F, B_0, Δ, D satisfy all of these conditions: $F \neq 0$ is weighted homogeneous of degree α and $F(x) = 0$; every variable X_i of F has $w_i < \alpha$ and zero constant Cantor coefficient; X_{B_0} has maximal weight; $\Delta + Dw_{B_0} = \alpha$; $\Delta \neq 0$ and every Cantor term of Δ is at least ω^β; the last Cantor term of w_{B_0} is below ω^β; $\lambda_0 < \alpha_{<\beta}$;

$$\alpha_1 \leq \alpha_{\geq\beta} \oplus \lambda_0, \qquad \alpha_1 \leq \alpha;$$

near 0, every translated truncation of $F(b)$ has degree below α_1; and every convolution term ρ using at least two translated truncations satisfies $\rho < \alpha_{\geq\beta} \oplus \lambda_0$.

The two required conclusions are:

1. $D = 1$. 2. If $X_{B'}$ occurs in F and $(w_{B'})_{<\beta} \oplus \eta = \lambda_0$ for some η, then there are a finite set S and polynomials U_B such that

$$\partial_{B'} F = \sum_{B \in S} (\partial_B F) U_B, \qquad \partial_{B_0} U_B = 0 \quad (B \in S),$$

and every $B \in S$ occurs in F and satisfies either $(w_B)_{<\beta} = \alpha_{<\beta}$, or

$$0 \neq (w_B)_{<\beta} \neq \alpha_{<\beta}, \qquad \nexists \eta,\ (w_B)_{<\beta} \oplus \eta = \lambda_0, \qquad w_{B'} < w_B.$$

Then evaluation at x_i is injective on the weighted-homogeneous polynomials of degree α.

Proof. By Theorem 43 (*Multiplicativity of the Cantor–Bendixson value*), the Cantor–Bendixson degree defines the associated graded ring in which injectivity is tested.

Assume a nonzero homogeneous relation F of degree α and choose a variable X_{B_0} of maximal weight. Write F as a polynomial in X_{B_0}. By Lemma 7 (*Weights of variables in a homogeneous relation*), every variable of F has weight below α. If the degree Δ of the leading coefficient is zero, or its last Cantor term is no later than that of w_{B_0}, the first hypothesis is contradictory.

Otherwise write the last Cantor term of Δ as ω^β. The evaluated relation has degree below α, so its translated truncations have degrees bounded by some $\alpha_1 < \alpha$. The part $\alpha_{<\beta}$ is nonzero and has zero constant Cantor coefficient. By Lemma 23 (*Weighted-degree bound for terms with two translated truncations*), there is a uniform bound. Enlarging it to include the part of α_1 below ω^β supplies $\lambda_0 < \alpha_{<\beta}$ and the required bound on every nonlinear convolution term.

The two remaining hypotheses now make X_{B_0} occur linearly and supply the partial-derivative identities. Lemma 11 (*Partial-derivative obstruction to a linear variable*) excludes this case as well. Hence no nonzero relation of degree α exists. □

`HahnSeries.Germ.injectiveAt_of_limit`

Lemma 60 (Simultaneous Cantor–Bendixson derivatives of homogeneous tuples)**.** *Let B be finite, let λ_b be ordinal weights, and let (u_b) be homogeneous of total degree d, where the constant Cantor coefficient of d is positive. There are a set $S \subseteq G$ and functions $D_b : G \to \mathrm{gr}_\nu$ such that:*

(i) the derivative of u_b agrees near 0 with D_b;

(ii) for every γ, $(D_b(\gamma))_b$ is homogeneous of total degree d', where $d' + 1 = d$;

(iii) every D_b vanishes outside S;

(iv) for every homogeneous function $a : G \to \mathrm{gr}_\nu$ of degree ρ, some homogeneous class of degree $\rho + 1$ has derivative equal near 0 to a on S and to zero outside S.

Proof. Choose a series representative for each active coordinate of the tuple. Its derivative is supported near 0 on one exact-rank set. By Lemma 48 (*Discrete finite unions of Cantor–Bendixson rank sets*), after one common negative cutoff the union S of these finitely many sets is discrete. The pointwise derivative tuple has predecessor degree d' and vanishes outside S; the operator used here is a derivation by Lemma 54 (*The Leibniz rule for the Cantor–Bendixson derivation*). The union is partially well ordered and agrees near 0 with its closure, so Lemma 57 (*Prescribing the Cantor–Bendixson derivative on a discrete set*) realizes every homogeneous prescription on S in degree one higher. □

`HahnSeries.Nonpositive.hasSyzygyIntegration`

Lemma 61 (Successor step for Cantor–Bendixson homogeneous evaluation)**.** *Let K be a field of characteristic zero, and let G be a nontrivial ordered abelian group with compatible additive uniformity and order topology. Assume that G is Cauchy complete, and let ν be the Cantor–Bendixson degree on $K((G^{\leq 0}))$. Let (x_i) be a minimal homogeneous generating system of gr_ν, with weights w_i. If the coefficient of $1 = \omega^0$ in the Cantor normal form of δ is positive and homogeneous evaluation at (x_i) is injective in every degree below δ, then it is injective in degree δ.*

Proof. Suppose a nonzero weighted-homogeneous polynomial F of degree δ evaluates to zero. The chain rule for the Cantor–Bendixson lowering derivation makes the derivative values of each relevant partial derivative pointwise combinations of the distinguished partial derivatives. By Lemma 56 (*The derivative criterion for successor ideal membership*), these pointwise combinations give ideal membership. Then Lemma 14 (*Weighted Euler identity*) places F in that ideal.

By Lemma 9 (*Partial-derivative decomposition of a homogeneous ideal relation*), choose a finite family of relevant partial derivatives. Their polynomial syzygies have a finite generating family by Lemma 15 (*Finite generation of polynomial syzygies*). Homogeneous ideal decomposition expresses every remaining partial derivative in the chosen family, producing a homogeneous coefficient tuple whose evaluation is a syzygy. Lemma 60 (*Simultaneous Cantor–Bendixson derivatives of homogeneous tuples*) supplies representatives that define a polynomial derivation. By Lemma 13 (*Weighted degree of polynomial derivations*), it lowers the constant Cantor coefficient. Apply the induction hypothesis to the resulting lower-degree syzygies, express them in the finite generating family, and choose homogeneous polynomial representatives by Lemma 4 (*Weighted-homogeneous polynomial representatives*). This expresses the evaluated cofactor tuple as a combination of evaluated syzygies. Consequently its linear part lies in the square of the positive-degree ideal, contradicting the defining independence of a minimal homogeneous generating system modulo that square. □

`HahnSeries.Nonpositive.injectiveAt_of_forall_lt`

Lemma 62 (Degreewise injectivity of homogeneous evaluation)**.** *Let K be a field of characteristic zero and G a nontrivial complete ordered abelian group with compatible additive uniformity and order topology. Let ν be the Cantor–Bendixson degree on $K((G^{\le 0}))$. Let $x_i \in (\mathrm{gr}_\nu)_{w_i}$ be a minimal homogeneous generating system, and choose series b_i representing x_i in degree w_i. Suppose*

$$\nu(b_i) \le w_i, \qquad \nu(b_i^{|y}) < w_i \quad (y < 0).$$

For every degree α, assume evaluation is injective in all degrees below α. Consider any data

$$0 \ne F \in K[X_i : i \in I], \quad B_0 \in \mathrm{vars}(F), \quad \beta, \Delta, \lambda_0, \alpha_1,$$

satisfying all of the following conditions:
** F is weighted homogeneous of degree α, $F(x) = 0$, every variable X_i of F has $w_i < \alpha$ and zero constant Cantor coefficient, and $w_i \le w_{B_0}$; * with $D = \deg_{B_0} F$, one has $\Delta \oplus D w_{B_0} = \alpha$, $\Delta \ne 0$, every Cantor term of Δ is at least ω^β, and the last Cantor term of w_{B_0} is below ω^β; * $\lambda_0 < \alpha_{<\beta}$ and*

$$\alpha_1 \le \alpha_{\ge\beta} \oplus \lambda_0, \qquad \alpha_1 \le \alpha;$$

** for all $y < 0$ sufficiently close to 0, the translated truncation of $F(b)$ at y has degree below α_1, and every convolution term ρ from a monomial of F using at least two translated truncations satisfies $\rho < \alpha_{\ge\beta} \oplus \lambda_0$.*

Suppose first that $D = 1$ for every such choice of data. Suppose also that whenever $X_{B'}$ occurs in F and $(w_{B'})_{<\beta} \oplus \eta = \lambda_0$ for some η, there are a finite set S and polynomials U_B for $B \in S$ such that

$$\frac{\partial F}{\partial X_{B'}} = \sum_{B \in S} \frac{\partial F}{\partial X_B} U_B, \qquad \frac{\partial U_B}{\partial X_{B_0}} = 0.$$

Every $B \in S$ occurs in F and satisfies either $(w_B)_{<\beta} = \alpha_{<\beta}$, or

$$0 \ne (w_B)_{<\beta} \ne \alpha_{<\beta}, \qquad \nexists \xi,\ (w_B)_{<\beta} \oplus \xi = \lambda_0, \qquad w_{B'} < w_B.$$

Then, for every α, a weighted-homogeneous polynomial P of degree α with $P(x) = 0$ is zero.

Proof. Proceed by transfinite induction on α. Degree zero contains only scalars because every w_i is positive.

If the constant Cantor coefficient of α is positive, Lemma 61 (*Successor step for Cantor–Bendixson homogeneous evaluation*) gives injectivity directly from the induction hypothesis.

It remains to consider $\alpha \neq 0$ with zero constant Cantor coefficient. For the representatives b_i, the translated-truncation hypothesis and Lemma 58 (*Leading-coefficient obstruction for homogeneous relations*) exclude the case in which the leading coefficient in a maximal variable is scalar or has no smaller last Cantor term. In the complementary case, the assumed equation $D = 1$ and the partial-derivative decompositions are exactly the remaining hypotheses of Lemma 59 (*Injectivity of evaluation when the degree is a limit ordinal*), which gives injectivity in degree α. □

`HahnSeries.Germ.injectiveAt_of_isMinimalSystem_of_limitOrdinalCases`

Theorem 63 (Algebraic independence of a minimal homogeneous generating system)**.** *Let $K, G, \nu, (x_i), (w_i)$, and the representatives (b_i) satisfy the ambient, minimality, representation, and translated-truncation hypotheses of Lemma 62 (*Degreewise injectivity of homogeneous evaluation*). Assume, conditionally on injectivity in all degrees below α, both conclusions required there for every tuple satisfying its relation, cutoff, translated-truncation, and convolution hypotheses: the maximal variable occurs with degree one, and each indicated partial derivative has the displayed finite decomposition with coefficients independent of X_{B_0} and with the displayed restrictions on the indices. Then (x_i) is algebraically independent over K.*

Proof. By Lemma 62 (*Degreewise injectivity of homogeneous evaluation*), evaluation at (x_i) is injective on weighted-homogeneous polynomials of every degree. If a polynomial P evaluates to zero, then its evaluated component in degree α is the evaluation of the weighted-homogeneous component P_α. Hence every P_α is zero. Since P is the finite sum of these components, $P = 0$. Thus polynomial evaluation at (x_i) is injective, which is equivalent to algebraic independence over K. □

`HahnSeries.Germ.algebraicIndependent_of_isMinimalSystem_of_limitOrdinalCases`

Lemma 64 (Local Jacobian ideal membership for translated partial derivatives)**.** *Let F be weighted homogeneous of degree α, evaluated at series b_i representing a minimal homogeneous generating system of weights w_i. Assume*

$$\deg(b_i) \leq w_i, \qquad \deg(b_i^{|y}) < w_i \quad \textit{for every } y < 0.$$

Assume evaluation is injective below α, every variable of F has weight below α, and at a negative cutoff γ the translated truncation of $F(b)$ has degree less than α'. Suppose also that every term in the translated-truncation expansion containing at least two proper truncations has degree less than α'', where $\alpha' \leq \alpha''$ and $\alpha' \leq \alpha$.

If $\alpha'' \leq \tau \oplus w_{B'}$, then the terms of weighted degree at least τ in a polynomial representing $(\partial_{B'} F)(b)^{|\gamma}$ belong to the ideal generated by $\partial_B F$ for the variables satisfying $w_{B'} < w_B$.

Proof. By Theorem 43 (*Multiplicativity of the Cantor–Bendixson value*), the Cantor–Bendixson degree defines the associated graded ring and its homogeneous components. By Lemma 4 (*Weighted-homogeneous polynomial representatives*), translated truncations have homogeneous polynomial representatives. Differentiate the translated-truncation Leibniz expansion. The derivative of the represented value of F, and both remainders containing at least two proper truncations, have no terms of degree at least τ. By Lemma 12 (*Vanishing criterion for partial derivatives of weighted-homogeneous polynomials*), every term indexed by $w_B \leq w_{B'}$ vanishes. The remaining first-order terms are multiples of $\partial_B F$ with $w_{B'} < w_B$. □

Lemma 65 (Maximal-variable linearity for the Cantor–Bendixson degree)**.** *Let K be a field of characteristic zero and G a nontrivial complete ordered abelian group with compatible additive uniformity and order topology. Let x_i be a minimal homogeneous generating system for the associated graded ring of the degree filtration, of weights w_i. Choose series b_i representing x_i such that*

$$\deg(b_i) \le w_i, \qquad \deg(b_i^{|y}) < w_i \quad \textit{for every } y < 0.$$

Assume evaluation at x_i is injective in every degree below α.

Let $0 \neq F \in K[X_i : i \in I]$ be weighted homogeneous of degree α with $F(x) = 0$. Suppose every variable of F has weight below α and zero constant Cantor coefficient. Choose X_{B_0} of maximal weight among the variables of F, and put $D = \deg_{X_{B_0}} F$. Suppose there are ordinals $\beta, \Delta, \lambda_0, \alpha_1$ such that

$$\Delta + Dw_{B_0} = \alpha, \qquad \Delta \neq 0, \qquad \lambda_0 < \alpha_{<\beta},$$

every Cantor term of Δ is at least ω^β, and the last Cantor term of w_{B_0} is below ω^β. Assume also

$$\alpha_1 \le \alpha_{\ge\beta} \oplus \lambda_0, \qquad \alpha_1 \le \alpha,$$

that near 0 every translated truncation of $F(b)$ has degree below α_1, and that every convolution term ρ using at least two translated truncations satisfies $\rho < \alpha_{\ge\beta} \oplus \lambda_0$.

Then $D = 1$.

Proof. Suppose $D \ge 2$ and put $\Theta = \partial_{B_0} F$. Then $\Theta \neq 0$ is homogeneous of degree $h = \Delta \oplus (D-1)w_{B_0} < \alpha$. Injectivity below α makes $\Theta(b)$ have degree exactly h.

Put $t = (w_{B_0})_{<\beta}$. The hypotheses give $\alpha_{<\beta} = Dt$ and $\lambda_0 \oplus 1 < (D-1)t \oplus t$. By Lemma 21 (*Cofinality below a Hessenberg sum*), choose $s < (D-1)t$ with $\lambda_0 \oplus 1 \le s \oplus t$, and set $\tau = h_{\ge\beta} \oplus s < h$.

By Lemma 64 (*Local Jacobian ideal membership for translated partial derivatives*), every component of a sufficiently late polynomial representative of $\Theta(b)^{|\gamma}$ of degree at least τ lies in the ideal generated by variables of F whose weights are strictly larger than w_{B_0}. Maximality of B_0 makes this ideal zero. But the translated truncation of the degree-h series $\Theta(b)$ has degree at least τ, so its representative has a nonzero component in that range, a contradiction. □

Lemma 66 (Partial-derivative decomposition when the degree is a limit ordinal)**.** *Let K be a field of characteristic zero and let G be a nontrivial, densely ordered abelian group without endpoints, equipped with a compatible additive uniformity and order topology. Assume that G is Cauchy complete. Let $(U_i)_{i\in J}$ be a decreasing family of open convex additive subgroups, indexed by a well-founded linear order, such that for every $\varepsilon > 0$ some U_i is contained in $(-\varepsilon, \varepsilon)$.*

Let x_i be a minimal homogeneous generating system for the associated graded ring of the degree filtration, of weights w_i. Choose series b_i representing x_i such that

$$\deg(b_i) \le w_i, \qquad \deg(b_i^{|y}) < w_i \quad \textit{for every } y < 0.$$

Assume evaluation at x_i is injective in every degree below α.

Let $0 \neq F \in K[X_i : i \in I]$ be weighted homogeneous of degree α with $F(x) = 0$. Suppose every variable of F has weight below α and zero constant Cantor coefficient. Choose X_{B_0} of maximal weight and put $D = \deg_{X_{B_0}} F$. Suppose there are ordinals $\beta, \Delta, \lambda_0, \alpha_1$ such that

$$\Delta + Dw_{B_0} = \alpha, \quad \Delta \neq 0, \quad \lambda_0 < \alpha_{<\beta}, \quad \alpha_1 \le \alpha, \quad \alpha_1 \le \alpha_{\ge\beta} \oplus \lambda_0,$$

every Cantor term of Δ is at least ω^β, and the last Cantor term of w_{B_0} is below ω^β. Assume that near 0 every translated truncation of $F(b)$ has degree below α_1, and every convolution term ρ using at least two translated truncations satisfies $\rho < \alpha_{\geq\beta} \oplus \lambda_0$.

If $X_{B'}$ occurs in F and $(w_{B'})_{<\beta} \oplus \eta = \lambda_0$ for some ordinal η, then there are a finite set S and polynomials U_B such that

$$\partial_{B'} F = \sum_{B\in S} (\partial_B F) U_B, \qquad \partial_{B_0} U_B = 0 \quad (B \in S).$$

Every $B \in S$ occurs in F and satisfies either $(w_B)_{<\beta} = \alpha_{<\beta}$, or

$$0 \neq (w_B)_{<\beta} \neq \alpha_{<\beta}, \qquad \nexists \xi,\ (w_B)_{<\beta} \oplus \xi = \lambda_0, \qquad w_{B'} < w_B.$$

Proof. Use strong induction on the number of variables of F with weight strictly larger than $w_{B'}$. Put $\Theta = \partial_{B'} F$. If $\Theta = 0$ there is nothing to prove. Otherwise Θ is homogeneous of a degree h with $h \oplus w_{B'} = \alpha$. Write $(w_{B'})_{<\beta} \oplus \eta = \lambda_0$ and set $\tau = h_{\geq\beta} \oplus \eta$.

By Lemma 64 (*Local Jacobian ideal membership for translated partial derivatives*), the components in degrees at least τ of sufficiently late translated truncations of $\Theta(b)$ lie in the ideal generated by the partial derivatives at heavier variables. By the induction hypothesis, every such intervening derivative is already generated by the required ones.

The ordinal window needed by the local-to-global step follows from Lemma 26 (*Hessenberg-sum decomposition with equal last Cantor terms*) and Lemma 25 (*Hessenberg-sum decomposition with unequal last Cantor terms*). In the unequal-last-term case, the Hessenberg-sum decomposition first identifies the upper Cantor terms, which supplies the required upper-term equality.

By Lemma 52 (*Local-to-global principle for homogeneous ideal membership*), the eventual componentwise membership becomes an exact polynomial identity with cofactors of the complementary weighted degrees. The coefficient of $\partial_B F$ has weighted degree strictly below w_{B_0}, so it is independent of X_{B_0}. This gives $\partial_{B_0} U_B = 0$ and completes the induction. □

`HahnSeries.Germ.LimitOrdinalRelationAtCutoff.exists_finset_pderiv_eq_sum_of_lowDegreePartAlgebraicLE`

Lemma 67 (Well-founded Archimedean-ball bases)**.** *Let G be an ordered topological abelian group. Suppose that the nonzero Archimedean classes of G have no least element in the magnitude order. Then there are a well-founded linear order I and a decreasing family $(U_i)_{i\in I}$ of open order-convex additive subgroups of G such that every symmetric open interval about 0 contains some U_i.*

Proof. Well-order the finite Archimedean classes arbitrarily and retain each class that is larger in the Archimedean-class order than every earlier class. These record classes are cofinal, while their inherited order is a subrelation of the chosen well-order and is therefore well-founded. Associate to each record class its Archimedean ball. The balls decrease with the record classes and are order-convex. The absence of a least nonzero magnitude makes every ball open, and cofinality of the record classes puts one inside every symmetric interval about 0. □

`ArchimedeanClass.exists_wellFounded_archimedeanBall_basis`

Theorem 68 (Algebraic independence in the Cantor–Bendixson associated graded ring)**.** *Let K be a field of characteristic zero and let G be a nontrivial, densely ordered abelian group without endpoints, equipped with a compatible additive uniformity and order topology. Assume that G is*

Cauchy complete. Suppose the nonzero Archimedean classes of G have no least element in the magnitude order. Let ν be the Cantor–Bendixson degree on $K((G^{\le 0}))$.

If $x_i \in (\mathrm{gr}_\nu)_{w_i}$ is a minimal homogeneous generating system and each x_i is represented by a series b_i such that

$$\nu(b_i) \le w_i, \qquad \nu(b_i^{|y}) < w_i \quad (y < 0),$$

then the family (x_i) is algebraically independent over K.

Proof. By Lemma 62 (*Degreewise injectivity of homogeneous evaluation*), injectivity of homogeneous evaluation reduces to the two nontrivial requirements when the degree is a limit ordinal. These requirements follow from Lemma 65 (*Maximal-variable linearity for the Cantor–Bendixson degree*) and Lemma 66 (*Partial-derivative decomposition when the degree is a limit ordinal*). By Lemma 67 (*Well-founded Archimedean-ball bases*), the hypothesis on Archimedean classes supplies a decreasing well-founded neighbourhood basis of open order-convex additive subgroups. Evaluation is therefore injective in every homogeneous degree, hence on the whole polynomial ring. ☐

`HahnSeries.Germ.algebraicIndependent_of_minimal_system`

5 Translated truncations

The test from the last chapter says which local identities would rule out a polynomial relation. This chapter gets those identities from the series. A translated truncation shows the part of a series just below a chosen exponent. When we truncate a polynomial expression in principal series, the convolution formula splits the answer according to how many factors were truncated. Working modulo the ideal of series bounded away from zero makes the leading degree of this expansion visible.

The terms with exactly one truncated factor form the expected partial derivative; this is the Leibniz rule in this setting. Terms with two or more truncated factors have smaller degree by the earlier ordinal estimates. A bound on the truncations of a relation can therefore be turned into membership in its Jacobian ideal. This gives the local calculation used at successor degrees. A limit degree has no single previous layer, so one more argument is needed.

Lemma 69 (Translated truncations of successor ordinal value)**.** *Let $\beta < \omega_1$ and $b \in K((\mathbb{R}^{\le 0}))$. If $v_J(b) = \omega^{\beta+1}$, then for every $\theta < 0$ there is $\gamma \in (\theta, 0)$ such that $v_J(b^{|\gamma}) = \omega^\beta$.*

Proof. By Fact 31 (*Support-tail characterization of the ordinal value (Ber00, Definition 5.2)*), choose above θ a support tail B of order type $\omega^{\beta+1} = \omega^\beta \cdot \omega$. Let S be its initial block of order type ω^β, and put $\gamma = \sup S$.

If $\beta = 0$, then $S = \{\gamma\}$. The translated truncation has a non-zero constant coefficient and a gap immediately below zero, so its ordinal value is 1.

If $\beta > 0$, then ω^β is an additively principal limit ordinal. The set S is a final segment of the support below γ, giving $v_J(b^{|\gamma}) \le \omega^\beta$. Every interval immediately below γ contains a final segment of S, still of order type ω^β, giving the reverse inequality. Thus $v_J(b^{|\gamma}) = \omega^\beta$, and the choice of B gives $\theta < \gamma < 0$. ☐

`Berarducci.exists_ordinalValue_translatedTruncation_eq_wpow_of_ordinalValue_eq_wpow_add_one`

Proposition 70 (Injectivity of the translated-truncation map on P_α)**.** *Let K be a field, let $\alpha < \omega_1$ be a successor ordinal, and write $\alpha = \beta + 1$. Put*

$$\mathrm{P}_\delta := J_{\omega^{\delta+1}} / J_{\omega^\delta}.$$

Let $\mathrm{Fun}_{0^-}(V)$ be the space of V-valued functions on intervals $(\eta, 0)$, identified when they agree sufficiently close to 0. Then the K-linear map

$$\partial_\alpha : \mathrm{P}_\alpha \longrightarrow \mathrm{Fun}_{0^-}(\mathrm{P}_\beta), \qquad \partial_\alpha([b]) = [\gamma \mapsto [b^{|\gamma}]],$$

is injective. The inner class is taken modulo J_{ω^β}.

Proof. By Lemma 35 (*Decrease of the ordinal value under translated truncation*), every series of ordinal value below ω^α has translated truncations of ordinal value below ω^β sufficiently close to 0. Thus translated truncation induces the displayed map on the quotient.

A nonzero class $[b] \in \mathrm{P}_\alpha$ has a representative with $v_J(b) = \omega^\alpha$. By Lemma 69 (*Translated truncations of successor ordinal value*), every interval $(\theta, 0)$ contains a cutoff γ with $v_J(b^{|\gamma}) = \omega^\beta$. Hence $[b^{|\gamma}] \neq 0$ in P_β, so $\partial_\alpha([b]) \neq 0$. The kernel of the linear map ∂_α is therefore zero, and the map is injective. ☐

`Berarducci.principalComponentDerivAt_injective`

Lemma 71 (Initial form of a weighted-homogeneous evaluation)**.** *Let K be a field. For each $i \in I$, let $x_i \in \mathrm{P}_{\alpha_i} \subseteq \widehat{\mathrm{P}}$ and choose $b_i \in K((\mathbb{R}^{\leq 0}))$ representing x_i, so that*

$$v_J(b_i) < \omega^{\alpha_i+1}, \qquad b_i + J_{\omega^{\alpha_i}} = x_i.$$

If $F \in K[X_i : i \in I]$ is weighted-homogeneous of degree β for the weights α_i, then

$$v_J(F(b_i)) < \omega^{\beta+1}, \qquad F(b_i) + J_{\omega^\beta} = F(x_i) \in \mathrm{P}_\beta.$$

Proof. Constants represent their images in degree 0, and representatives are preserved by addition, multiplication, and powers. Hence every monomial in $F(b_i)$ represents the corresponding monomial in $F(x_i)$ in its weighted degree. Since every monomial of F has weighted degree β, summing gives the two asserted properties in degree β. ☐

`Berarducci.Lifts.aeval_represents`

Proposition 72 (Existence of polynomial representatives modulo J)**.** *Let K be a field, let $(x_i)_{i \in I}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}} = \bigoplus_{\beta<\omega_1} \mathrm{P}_\beta$, with $x_i \in \mathrm{P}_{w_i}$, and choose representatives b_i satisfying*

$$v_J(b_i) < \omega^{w_i+1}, \qquad b_i + J_{\omega^{w_i}} = x_i.$$

If $\alpha < \omega_1$ and $u \in K((\mathbb{R}^{\leq 0}))$ satisfies $v_J(u) < \omega^\alpha$, then there is a polynomial $F \in K[X_i : i \in I]$ such that every monomial of F has weighted degree less than α and

$$u \equiv F(b_i) \pmod{J}.$$

Proof. Use well-founded induction on $v_J(u)$. The case $u \in J$ is represented by 0. Otherwise $v_J(u) = \omega^\beta$ for some $\beta < \alpha$. Since the chosen b_i are available by Fact 30 (*Principal representatives of* P_α *(LM24, Remark 7.2.4)*), and since the x_i generate $\widehat{\mathrm{P}}$, Lemma 4 (*Weighted-homogeneous polynomial representatives*) gives a weighted-homogeneous polynomial F_0 of degree β whose value $F_0(x_i)$ is the class of u in P_β. By Lemma 71 (*Initial form of a weighted-homogeneous evaluation*), $F_0(b_i)$ represents the same class, so

$$v_J\big(u - F_0(b_i)\big) < v_J(u).$$

Apply the induction hypothesis to this difference, obtaining F_1, and take $F = F_0 + F_1$. $\square$

`Berarducci.Lifts.exists_degreeLT_toGerm_aeval_eq`

Proposition 73 (Ordinal value of a polynomial evaluation)**.** *Let K be a field. For each $i \in I$, let $x_i \in \mathrm{P}_{w_i} \subseteq \widehat{\mathrm{P}}$ and choose a representative $b_i \in K((\mathbb{R}^{\le 0}))$. Let $F \in K[X_i : i \in I]$ be nonzero, and let $\deg_w(F)$ be the largest weighted degree of a monomial of F. If evaluation at (x_i) is injective on the weighted-homogeneous polynomials of degree $\deg_w(F)$, then*

$$v_J(F(b_i)) = \omega^{\deg_w(F)}.$$

Proof. Let F_d be the weighted-homogeneous component of F of top degree $d = \deg_w(F)$. It is nonzero, and injectivity in degree d gives $F_d(x_i) \neq 0$. By Lemma 71 (*Initial form of a weighted-homogeneous evaluation*), $F_d(b_i)$ represents this nonzero class in P_d, so $v_J(F_d(b_i)) = \omega^d$. Every monomial of $F - F_d$ has weighted degree less than d, whence $v_J((F - F_d)(b_i)) < \omega^d$. Since $F(b_i) = F_d(b_i) + (F - F_d)(b_i)$, the strict inequality and the ultrametric property give $v_J(F(b_i)) = \omega^d$. $\square$

`Berarducci.Lifts.ordinalValue_aeval_eq_wpow_weightedTotalDegree`

Lemma 74 (Omission of a maximal-weight variable from proper truncation representatives)**.** *Let K be a field, let $(x_i)_{i\in I}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$ with $x_i \in \mathrm{P}_{w_i}$, and choose representatives $b_i \in K((\mathbb{R}^{\le 0}))$. Fix $\alpha < \omega_1$, and suppose that evaluation at (x_i) is injective on weighted-homogeneous polynomials of every degree below α. For $v_J(u) < \omega^\alpha$, denote by P_u the unique polynomial whose monomials have weight below α and for which $P_u(b_i) \equiv u$ $(\mathrm{mod}\ J)$.*

If $B_0 \in I$ and $w_{B_0} < \alpha$, then there is $\varepsilon > 0$ such that

$$X_{B_0} \notin \mathrm{vars}(P_{b_{B_0}^{|\gamma}}) \qquad (-\varepsilon < \gamma < 0).$$

Proof. Since b_{B_0} represents $x_{B_0} \in \mathrm{P}_{w_{B_0}}$, its ordinal value is below $\omega^{w_{B_0}+1}$. By Lemma 35 (*Decrease of the ordinal value under translated truncation*), for every $\gamma < 0$ sufficiently close to 0,

$$v_J(b_{B_0}^{|\gamma}) < \omega^{w_{B_0}}.$$

Apply Proposition 72 (*Existence of polynomial representatives modulo J*) at w_{B_0}. Its polynomial has every monomial of weight below w_{B_0}, and uniqueness from Proposition 73 (*Ordinal value of a polynomial evaluation*) identifies it with the representative $P_{b_{B_0}^{|\gamma}}$ chosen below α. A monomial involving X_{B_0} has weight at least w_{B_0}, so no such monomial can occur. $\square$

`Berarducci.Lifts.exists_forall_pol_translatedTruncation_lift_mem`

Proposition 75 (Ordinal value of a polynomial representative)**.** *Let K be a field, let $(x_i)_{i\in I}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$ with $x_i \in \mathrm{P}_{w_i}$, and choose representatives b_i. Fix $\alpha < \omega_1$ and assume that evaluation at (x_i) is injective on weighted-homogeneous polynomials of every degree less than α.*

For $u \in K((\mathbb{R}^{\leq 0}))$ with $v_J(u) < \omega^\alpha$, let P_u be the unique polynomial whose monomials have weight less than α and which satisfies $P_u(b_i) \equiv u \pmod J$. If $u \notin J$, then

$$v_J(u) = \omega^{\deg_w(P_u)},$$

where $\deg_w(P_u)$ is the largest weighted degree of a monomial of P_u.

Proof. Existence and congruence of P_u come from Proposition 72 (*Existence of polynomial representatives modulo J*). The polynomial is nonzero, since otherwise its congruence would put u in J. Its largest weighted degree is less than α, so the injectivity hypothesis applies there. By Proposition 73 (*Ordinal value of a polynomial evaluation*),

$$v_J(P_u(b_i)) = \omega^{\deg_w(P_u)}.$$

Congruence modulo J preserves every nonzero ordinal value, giving the asserted equality for u. □

`Berarducci.Lifts.ordinalValue_eq_wpow_weightedTotalDegree_pol`

Theorem 76 (Leibniz rule for the lowering derivation)**.** *Let K be a field. On each successor component $\mathrm{P}_{\alpha+1} \subseteq \widehat{\mathrm{P}}$, let ∂ send a class represented by b to the germ at 0^- of*

$$\gamma \longmapsto b^{|\gamma} + J_{\omega^\alpha} \in \mathrm{P}_\alpha,$$

and let ∂ vanish on P_0 and on components of limit degree. Extend this map K-linearly to

$$\partial : \widehat{\mathrm{P}} \longrightarrow \mathrm{Fun}_{0^-}(\widehat{\mathrm{P}}).$$

Then, for all $B, C \in \widehat{\mathrm{P}}$,

$$\partial(BC) = \partial(B)C + B\partial(C)$$

in $\mathrm{Fun}_{0^-}(\widehat{\mathrm{P}})$.

Proof. First suppose B and C are homogeneous. Degree-zero components are scalars, so the identity follows from K-linearity. For positive degrees, use Fact 30 (*Principal representatives of P_α (LM24, Remark 7.2.4)*) to choose representatives. If the degree of B is a successor, apply Lemma 29 (*Convolution formula for translated truncations*) to $(BC)^{|\gamma}$. The two boundary terms give $\partial(B)C + B\partial(C)$; by Lemma 35 (*Decrease of the ordinal value under translated truncation*), every interior term has smaller ordinal value and vanishes in the target component. The case where only the degree of C is a successor follows by commutativity. If both positive degrees are limits, their natural sum is a limit and all three derivatives vanish.

Finally decompose arbitrary B and C into their finite sums of homogeneous components. K-linearity of ∂ and distributivity extend the homogeneous identity to all of $\widehat{\mathrm{P}}$. □

`Berarducci.principalSubringDerivation_mul`

Lemma 77 (Local finiteness of translated truncations at successor ordinal value)**.** *Let K be a field, let $\delta < \omega_1$, and let $p \in K((\mathbb{R}^{\leq 0}))$ satisfy*

$$v_J(p) = \mathrm{ot}(p) = \omega^{\delta+1}.$$

For every $\xi_0 < 0$, the set

$$\{\xi \leq \xi_0 : v_J(p^{|\xi}) \geq \omega^\delta\}$$

is finite.

Proof. Otherwise choose a strictly increasing sequence (γ_n) in this set. The support between consecutive γ_n has order type at least ω^δ, so the support at or below ξ_0 has order type at least $\omega^\delta \cdot \omega = \omega^{\delta+1}$. Removing that part does not change the series modulo J, hence the remaining support above ξ_0 also has order type at least $\omega^{\delta+1}$. Splitting the support at ξ_0 would therefore give

$$\mathrm{ot}(p) \geq \omega^{\delta+1} + \omega^{\delta+1} > \omega^{\delta+1},$$

a contradiction. □

Berarducci.cutoffsGE_inter_Iic_finite_of_neg

Lemma 78 (Countable-support representatives of P_α)**.** *Let K be a field, let $\alpha = \beta + 1 < \omega_1$, and let $0 \neq x \in \mathrm{P}_\alpha$. Then x has a principal representative p of degree α. Let $D_p : \mathbb{R} \to \mathrm{P}_\beta$ be the function used to represent $\partial_\alpha(x)$: whenever $v_J(p^{|\xi}) < \omega^{\beta+1}$, put $D_p(\xi) = [p^{|\xi}]$, and put $D_p(\xi) = 0$ otherwise. There is a strictly increasing sequence of negative reals $(\gamma_k)_{k\in\mathbb{N}}$, cofinal in 0, such that*

$$\{\xi < 0 : D_p(\xi) \neq 0\} \subseteq \{\gamma_k : k \in \mathbb{N}\}.$$

Proof. By Fact 30 (*Principal representatives of* P_α *(LM24, Remark 7.2.4)*), choose a principal representative p of x of degree α. The set

$$Z := \{\xi < 0 : D_p(\xi) \neq 0\}$$

is well ordered because a decreasing sequence in Z would induce a decreasing sequence in $\mathrm{supp}(p)$. By Lemma 77 (*Local finiteness of translated truncations at successor ordinal value*), each initial segment of Z is finite. By Proposition 70 (*Injectivity of the translated-truncation map on* P_α), D_p cannot vanish throughout any interval $(\eta, 0)$, since that would give $\partial_\alpha(x) = 0$ and hence $x = 0$. Thus Z is infinite and cofinal in 0. Enumerating Z in increasing order gives the required sequence. □

Berarducci.exists_principal_representative_derivAt

Lemma 79 (Detection of support order type by a translated truncation)**.** *Let K be a field, let $b \in K((\mathbb{R}^{\leq 0}))$, and let $\rho < \omega_1$ be nonzero. If*

$$\mathrm{ot}(\mathrm{supp}(b)) \geq \omega^\rho,$$

then there are $\xi \leq 0$ and $y \in \mathrm{supp}(b)$ such that $y \leq \xi$ and

$$v_J(b^{|\xi}) \geq \omega^\rho.$$

Proof. Choose an initial segment B of $\mathrm{supp}(b)$ of order type ω^ρ, and let $\xi = \sup B$. Then $\xi \le 0$ and lies above every point of the nonempty set B.

Since $\rho \neq 0$, the ordinal ω^ρ is additively principal and at least ω. For every $\theta < \xi$, the part of B above θ still has order type ω^ρ. At most one of its points is at or above ξ, so its part in (θ, ξ) also has order type at least ω^ρ. Thus every support interval immediately below ξ has order type at least ω^ρ, which gives $v_J(b^{|\xi}) \ge \omega^\rho$. $\square$

`Berarducci.exists_le_wpow_le_ordinalValue_translatedTruncation_of_le_supportOrderType`

Lemma 80 (Detection of interval support order type by a translated truncation)**.** *Let K be a field, let $b \in K((\mathbb{R}^{\le 0}))$, let $c \in \mathbb{R}$, and let $\rho < \omega_1$ be nonzero. If*

$$\mathrm{supp}(b) \subseteq (c, 0], \qquad \mathrm{ot}(\mathrm{supp}(b)) \ge \omega^\rho,$$

then there is $\xi \in (c, 0]$ such that $v_J(b^{|\xi}) \ge \omega^\rho$.

Proof. Apply Lemma 79 (*Detection of support order type by a translated truncation*). Its cutoff ξ lies above a point $y \in \mathrm{supp}(b)$. Since $c < y \le \xi \le 0$, the same cutoff belongs to $(c, 0]$. $\square$

`Berarducci.exists_wpow_le_ordinalValue_translatedTruncation_of_le_supportOrderType`

Corollary 81 (Support-order bound from uniformly small translated truncations)**.** *Let K be a field, let $b \in K((\mathbb{R}^{\le 0}))$, and let $\rho < \omega_1$ be nonzero. If*

$$v_J(b^{|\xi}) < \omega^\rho \qquad \textit{for every } \xi \le 0,$$

then

$$\mathrm{ot}(\mathrm{supp}(b)) < \omega^\rho.$$

Proof. Otherwise Lemma 79 (*Detection of support order type by a translated truncation*) supplies some $\xi \le 0$ with $v_J(b^{|\xi}) \ge \omega^\rho$, contradicting the hypothesis. $\square$

`Berarducci.supportOrderType_lt_of_forall_ordinalValue_translatedTruncation_lt`

Lemma 82 (Ordinal-value bound for proper translated truncations of a principal series)**.** *Let K be a field and let $b \in K((\mathbb{R}^{\le 0}))$ be a principal series of degree $\sigma < \omega_1$. For every $\zeta < 0$,*

$$v_J(b^{|\zeta}) < \omega^\sigma.$$

Proof. The support of b has order type ω^σ and supremum 0. Choose $y \in \mathrm{supp}(b)$ with $\zeta < y < 0$. The support of the truncation $b_{|\zeta}$ embeds in the proper initial segment $\mathrm{supp}(b) \cap (-\infty, y)$, so its order type is strictly below ω^σ. Translation preserves order type, and the ordinal value of a series is at most the order type of its support. Hence $v_J(b^{|\zeta}) < \omega^\sigma$. $\square$

`Berarducci.ordinalValue_translatedTruncation_lt_of_isPrincipal`

Lemma 83 (Homogeneous component representing a class in P_β)**.** *Let K be a field, let $(x_i)_{i\in I}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$ with $x_i \in \mathrm{P}_{w_i}$, and choose representatives $b_i \in K((\mathbb{R}^{\le 0}))$. Fix $\alpha < \omega_1$, and suppose that evaluation at (x_i) is injective on weighted-homogeneous polynomials of every degree below α.*

For $v_J(u) < \omega^\alpha$, let P_u be the unique polynomial whose monomials have weight below α and for which $P_u(b_i) \equiv u \pmod{J}$. If $\beta < \alpha$ and $v_J(u) < \omega^{\beta+1}$, then the weighted-homogeneous component $(P_u)_\beta$ satisfies

$$(P_u)_\beta(x_i) = u + J_{\omega^\beta} \quad \textit{in } \mathrm{P}_\beta.$$

Proof. By Proposition 72 (*Existence of polynomial representatives modulo J*), choose P_u; then Proposition 73 (*Ordinal value of a polynomial evaluation*) makes it unique. Applying Proposition 72 (*Existence of polynomial representatives modulo J*) at $\beta + 1$ and using this uniqueness shows that every monomial of P_u has weight below $\beta + 1$. Therefore evaluation of its degree-β weighted-homogeneous component at (b_i) represents the degree-β class of $P_u(b_i)$. Finally $P_u(b_i) \equiv u \pmod{J}$, so the same component represents the class of u. □

`Berarducci.Lifts.represents_aeval_weightedHomogeneousComponent_pol`

Lemma 84 (Translated truncations of monomials in principal series)**.** *Let K be a field. For each $i \in I$, let $x_i \in \mathrm{P}_{w_i} \subseteq \widehat{\mathrm{P}}$ and choose a principal series b_i of degree w_i representing x_i, where $w_i \neq 0$. Then, for every monomial X^d and every $\zeta < 0$,*

$$v_J((X^d(b_i))^{|\zeta}) < \omega^{\mathrm{wt}(d)}.$$

Proof. Argue by strong induction on the number of factors of X^d. The constant monomial has zero translated truncation. Otherwise write $X^d = X^{d'} X_i$. By Lemma 71 (*Initial form of a weighted-homogeneous evaluation*), $X^{d'}(b_i)$ and b_i satisfy the ordinary ordinal-value bounds, and Lemma 29 (*Convolution formula for translated truncations*) expresses the translated truncation of their product as a finite sum modulo J.

Since $\zeta < 0$, each convolution summand contains a proper translated truncation. If its first cutoff is negative, the induction hypothesis lowers the first factor. At cutoff 0, the second factor is lowered by Lemma 82 (*Ordinal-value bound for proper translated truncations of a principal series*). Multiplicativity and strict monotonicity of Hessenberg's natural sum put every product below $\omega^{\mathrm{wt}(d)}$; the ultrametric inequality gives the same bound for the finite sum. □

`Berarducci.Lifts.IsPrincipal.ordinalValue_translatedTruncation_aeval_monomial_lt`

Lemma 85 (Translated truncations of weighted-homogeneous polynomials)**.** *Let K be a field. For each $i \in I$, let $x_i \in \mathrm{P}_{w_i} \subseteq \widehat{\mathrm{P}}$ and choose a principal series b_i of degree w_i representing x_i, where $w_i \neq 0$. If $Q \in K[X_i : i \in I]$ is weighted homogeneous of degree σ, then, for every $\zeta < 0$,*

$$v_J((Q(b_i))^{|\zeta}) < \omega^{\sigma}.$$

Proof. Expand Q over its finite monomial support. Every occurring monomial has weight σ, so Lemma 84 (*Translated truncations of monomials in principal series*) bounds the ordinal value of its translated truncation by ω^{σ}. Multiplication by its scalar coefficient preserves this bound, and the ultrametric inequality gives it for the finite sum. □

`Berarducci.Lifts.IsPrincipal.ordinalValue_translatedTruncation_aeval_lt`

Lemma 86 (High-degree components of translated truncations of a polynomial multiple)**.** *Let K be a field and let $\mathcal{B}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$, with principal representatives b_B. Assume evaluation is injective in every degree below α. Let $Q \in K[X_B : B \in \mathcal{B}]$ be weighted homogeneous of degree σ. Suppose*

$$v_J(u) < \omega^{\rho+1}, \qquad v_J(u^{|\zeta}) < \omega^{\rho} \quad (\zeta < 0),$$
$$\rho \oplus \sigma < \alpha, \qquad \rho \oplus \theta < \tau \quad (\theta < \sigma).$$

Then, for every $\xi \leq 0$,

$$\mathrm{pol}_{<\alpha}\Big((uQ(b_B))^{|\xi}\Big)_{\geq \tau} \in (Q) \subseteq K[X_B : B \in \mathcal{B}].$$

Proof. By Proposition 72 (*Existence of polynomial representatives modulo J*), translated truncations of ordinal value below ω^α have polynomial representatives of degree below α; Proposition 73 (*Ordinal value of a polynomial evaluation*) gives the needed uniqueness. Expand the translated truncation of $uQ(b_B)$ by convolution. The endpoint term is $\mathrm{pol}_{<\alpha}(u^{|\xi})Q$, so its components of degree at least τ lie in (Q). In every other term the $Q(b_B)$ factor is translated-truncated at a negative cutoff. By Lemma 85 (*Translated truncations of weighted-homogeneous polynomials*), its polynomial representative has degree below σ, while the representative of the u factor has degree at most ρ. Thus $\rho \oplus \theta < \tau$ makes the component of degree at least τ vanish. Summing proves the claim. $\square$

`Berarducci.Lifts.componentsGE_pol_translatedTruncation_mul_aeval_mem`

Proposition 87 (Prescribed P_δ classes at translated truncations)**.** *Let K be a field and let δ be a countable ordinal. Let $(\gamma_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of negative reals cofinal below 0, and let $a_k \in \mathrm{P}_\delta$ for every k. Then there is a series $s \in K((\mathbb{R}^{\leq 0}))$ such that*

$$v_J(s) < \omega^{\delta\oplus 1\oplus 1}.$$

For every k,

$$v_J(s^{|\gamma_k}) < \omega^{\delta\oplus 1}, \qquad s^{|\gamma_k} + J_{\omega^\delta} = a_k \quad \text{in } \mathrm{P}_\delta.$$

Moreover, if $\gamma_0 < \xi < 0$ and $\xi \neq \gamma_k$ for every k, then

$$v_J(s^{|\xi}) < \omega^\delta.$$

Proof. By Fact 30 (*Principal representatives of* P_α *(LM24, Remark 7.2.4)*), choose a principal series w_k representing each nonzero a_k, and choose $w_k = 0$ when $a_k = 0$. Each support has order type at most ω^δ. By Lemma 35 (*Decrease of the ordinal value under translated truncation*), choose $c_k < 0$ such that every proper translated truncation of w_k at a point of $(c_k, 0)$ has ordinal value below ω^δ; enlarge c_k if necessary so that $\gamma_{k-1} \leq \gamma_k + c_k$ for $k \geq 1$.

Place $(w_k)_{>c_k}$ in the interval $(\gamma_k + c_k, \gamma_k]$. These intervals are disjoint and strictly increasing. By Lemma 19 (*Order type of a countable ordered union*), their union is well ordered and has order type at most $\omega^{\delta\oplus 1}$. Hence the Hahn sum s exists and $v_J(s) < \omega^{\delta\oplus 1\oplus 1}$.

At γ_k, every earlier interval contributes only an element of J, the k-th interval has the same germ as w_k, and every later interval is excluded. Thus $s^{|\gamma_k}$ represents a_k. Between two prescribed cutoffs, its germ is either zero or the germ of a proper translated truncation of one w_k, whose ordinal value is below ω^δ by the choice of c_k. $\square$

`Berarducci.exists_sumAlongCutoffs`

Proposition 88 (Derivative criterion for homogeneous ideal membership)**.** *Let K be a field and let $(q_j)_{j\in I}$ be a finite family in $\widehat{\mathrm{P}}$. Suppose that q_j is homogeneous of degree c_j and that the coefficient of ω^0 in the Cantor normal form of c_j is zero for every j. Let α be an ordinal whose coefficient of ω^0 is positive, and let $x \in \mathrm{P}_\alpha$. If $f\colon \mathbb{R} \to \widehat{\mathrm{P}}$ satisfies*

$$f(\xi) \in (q_j : j \in I) \quad \text{for every } \xi \in \mathbb{R}$$

and the lowering derivative ∂x is the class of f at 0^-, then

$$x \in (q_j : j \in I) \subseteq \widehat{\mathrm{P}}.$$

Proof. By Lemma 78 (*Countable-support representatives of* P_α), choose a principal series representing x whose derivative is supported on a strictly increasing sequence of cutoffs cofinal below 0. At each cutoff, decompose the value of f separately as a homogeneous linear combination of the q_j. For each j, apply Proposition 87 (*Prescribed* P_δ *classes at translated truncations*) to the resulting sequence of coefficients, obtaining a homogeneous element w_j whose derivative has exactly those values at the chosen cutoffs and vanishes between them.

The coefficient condition on c_j gives $\partial q_j = 0$. Hence Theorem 76 (*Leibniz rule for the lowering derivation*) shows that

$$\partial\left(x - \sum_j q_j w_j\right) = 0:$$

at a chosen cutoff this is the selected homogeneous decomposition of $f(\xi)$, and between the cutoffs both representative functions vanish near 0. The expression in parentheses is homogeneous of degree α. The same injectivity of the lowering derivative used in the representative theorem applies because the coefficient of ω^0 in α is positive. Therefore $x = \sum_j q_j w_j$ and $x \in (q_j : j \in I)$. □

`Berarducci.mem_span_of_principalSubringDerivation_eq_coe`

Lemma 89 (Convolution formula for polynomial representatives)**.** *Let K be a field. Let $(x_i)_{i\in I}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$, with $x_i \in \mathrm{P}_{w_i}$, and choose series b_i representing the x_i. Assume that evaluation at (x_i) is injective in every weighted degree below $\alpha < \omega_1$. For every series a with $v_J(a) < \omega^\alpha$, write $\mathrm{pol}_{<\alpha}(a)$ for the unique polynomial whose monomials have weight below α and such that*

$$a \equiv \mathrm{pol}_{<\alpha}(a)(b_i) \pmod J.$$

Let $u, v \in K((\mathbb{R}^{\leq 0}))$ satisfy

$$v_J(u) < \omega^{\beta+1}, \qquad v_J(v) < \omega^{\beta'+1}, \qquad \beta, \beta' < \alpha, \qquad \beta \oplus \beta' \leq \alpha.$$

For $\gamma \in \mathbb{R}$, put

$$C_\gamma(u,v) = \{\xi \in \mathrm{cl}(\mathrm{supp}(u)) : \gamma - \xi \in \mathrm{cl}(\mathrm{supp}(v))\}.$$

Then there is $\varepsilon > 0$ such that, for every $\gamma \in (-\varepsilon, 0)$ and every finite S with $C_\gamma(u,v) \subseteq S \subseteq [\gamma, 0]$,

$$\mathrm{pol}_{<\alpha}((uv)^{|\gamma}) = \sum_{\xi\in S} \mathrm{pol}_{<\alpha}(u^{|\xi})\,\mathrm{pol}_{<\alpha}(v^{|\gamma-\xi}).$$

Proof. By Lemma 35 (*Decrease of the ordinal value under translated truncation*), choose one interval on which the translated truncations of u, v, and uv have the required lower ordinal values. By Proposition 72 (*Existence of polynomial representatives modulo J*) and Proposition 73 (*Ordinal value of a polynomial evaluation*), their degree-$<\alpha$ polynomial representatives exist and are unique.

Fix γ in this interval and $\xi \in S$. Since $\gamma < 0$ and $\gamma \leq \xi \leq 0$, at least one of ξ and $\gamma - \xi$ is negative. Its translated truncation has polynomial weight strictly below β or β', while the other has weight at most the corresponding ordinal. Hence the product has weight strictly below $\beta \oplus \beta' \leq \alpha$.

Lemma 29 (*Convolution formula for translated truncations*) gives the displayed identity modulo J over $C_\gamma(u,v)$. Every additional term indexed by S vanishes modulo J. Uniqueness of the polynomial representatives turns this congruence into the asserted polynomial identity. □

Lemma 90 (Leading coefficient after translated truncation). *Let $(x_i)_{i\in I}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$, with $x_i \in \mathrm{P}_{w_i}$, and choose series b_i representing the x_i. Assume that evaluation at (x_i) is injective in every weighted degree below $\alpha < \omega_1$, and write P_a for the resulting polynomial representative of a series a of ordinal value below ω^α.*

Fix $B_0 \in I$ with $w_{B_0} < \alpha$. Let $\beta < \alpha$ and let $u \in K((\mathbb{R}^{\le 0}))$ satisfy $v_J(u) < \omega^{\beta+1}$. Suppose that P_u does not involve X_{B_0} and that the same holds for $P_{u^{|\gamma}}$ for every $\gamma < 0$ sufficiently close to 0.

If $(e \odot w_{B_0}) \oplus \beta \le \alpha$, then, for every $\gamma < 0$ sufficiently close to 0, the polynomial $P_{(b_{B_0}^e u)^{|\gamma}}$ has degree at most e in X_{B_0} and

$$[X_{B_0}^e]\, P_{(b_{B_0}^e u)^{|\gamma}} = P_{u^{|\gamma}}.$$

Proof. Induct on e. The case $e = 0$ is the hypothesis on u. For the successor step, apply Lemma 89 (*Convolution formula for polynomial representatives*) to $b_{B_0}(b_{B_0}^e u)$. The summand at cutoff 0 is $X_{B_0} P_{(b_{B_0}^e u)^{|\gamma}}$, whose coefficients are controlled by the induction hypothesis.

In every other summand the first factor is a proper translated truncation of b_{B_0}, so its polynomial omits X_{B_0} by Lemma 74 (*Omission of a maximal-weight variable from proper truncation representatives*). At cutoff γ, the second translated truncation is $b_{B_0}^e u$, whose unique polynomial representative with monomial weights below α is $P_u X_{B_0}^e$; at the interior cutoffs the induction hypothesis bounds its X_{B_0}-degree by e. Thus only the cutoff-0 summand can contribute in degree $e+1$, and its coefficient there is $P_{u^{|\gamma}}$. □

Berarducci.Lifts.exists_forall_xCoeff_pol_translatedTruncation_pow_mul

Lemma 91 (Polynomial Leibniz rule for translated truncations). *Let K be a field, let $(x_i)_{i\in I}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$ with $x_i \in \mathrm{P}_{w_i}$, and choose series b_i representing the x_i. Assume that evaluation at (x_i) is injective in every weighted degree below $\alpha < \omega_1$. For a series a of ordinal value below ω^α, write P_a for its resulting polynomial representative.*

Let $G \in K[X_i : i \in I]$. Suppose every monomial X^d of G has weight at most α, and $w_i < \alpha$ for every variable X_i occurring in G. Then, for every $\gamma < 0$ sufficiently close to 0,

$$v_J((G(b_i))^{|\gamma}) < \omega^\alpha$$

and there is a polynomial R_γ such that

$$P_{(G(b_i))^{|\gamma}} = \sum_{i\in \mathrm{vars}(G)} P_{b_i^{|\gamma}} \frac{\partial G}{\partial X_i} + R_\gamma.$$

More precisely, for every monomial $X^{d'}$ of R_γ, there are a monomial X^d of G, an integer $k \ge 2$, a factorisation

$$X^d = X^{d_0} X_{i_1} \cdots X_{i_k},$$

with factors listed with multiplicity, and ordinals $\rho_r < w_{i_r}$ such that

$$\mathrm{wt}(d') = \mathrm{wt}(d_0) \oplus \rho_1 \oplus \cdots \oplus \rho_k.$$

Proof. First take $G = X^d$ and argue by strong induction on the number of factors; the constant case has zero translated truncations. The evaluation $X^d(b_i)$ represents a homogeneous element of weight $\mathrm{wt}(d)$, so its ordinal value is below $\omega^{\mathrm{wt}(d)+1}$. Sufficiently late translated truncations therefore have ordinal value below $\omega^{\mathrm{wt}(d)} \leq \omega^{\alpha}$.

In the nonconstant case, split off one factor X_i and apply Lemma 89 (*Convolution formula for polynomial representatives*). The two endpoint summands, after identifying the unique polynomial representatives of the untruncated factors, give the ordinary product rule. At every interior cutoff, the split-off lift is properly translated, so its representative has monomial weights strictly below w_i. The induction expansion of the other factor has already replaced at least one variable weight by a smaller ordinal. Their product therefore replaces at least two weights, and an inductive remainder remains of the stated form after multiplication by X_i.

Finally expand G over its finite monomial support, choose one common punctured interval, and sum the monomial identities. The ultrametric inequality gives the asserted ordinal-value bound. ☐

`Berarducci.Lifts.exists_forall_pol_translatedTruncation_aeval`

Lemma 92 (Differentiated polynomial Leibniz rule)**.** *Let K be a field, let $(x_i)_{i\in I}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$ with $x_i \in \mathrm{P}_{w_i}$, and choose series b_i representing the x_i. Assume that evaluation at (x_i) is injective in every weighted degree below $\alpha < \omega_1$, and write P_a for the polynomial representative of a series a of ordinal value below ω^{α}.*

Let $F \in K[X_i : i \in I]$. Suppose every monomial of F has weight at most α, and $w_i < \alpha$ for every variable X_i occurring in F. For every $v \in I$ and every $\gamma < 0$ sufficiently close to 0, there are polynomials R_γ, R'_γ such that

$$P_{((\partial_v F)(b_i))^{|\gamma}} = \partial_v P_{(F(b_i))^{|\gamma}} - \sum_{j\in \mathrm{vars}(F)} (\partial_v P_{b_j^{|\gamma}})\, \partial_j F - \partial_v R_\gamma + R'_\gamma,$$

where every monomial of R_γ, respectively R'_γ, is obtained from a monomial of F, respectively $\partial_v F$, by replacing the weights of at least two variable factors, counted with multiplicity, by strictly smaller ordinals.

Proof. Partial differentiation weakly decreases every monomial weight and introduces no new variables. Hence Lemma 91 (*Polynomial Leibniz rule for translated truncations*) applies to both F and $\partial_v F$ on one common punctured interval. Differentiate the expansion for F, use the polynomial product rule and commutativity of partial derivatives, and subtract the expansion for $\partial_v F$. Variables of F absent from $\partial_v F$ contribute zero, so both sums may be indexed by $\mathrm{vars}(F)$. Rearrangement gives the displayed identity while preserving the two remainder conditions. ☐

`Berarducci.Lifts.exists_forall_pol_translatedTruncation_aeval_pderiv`

Lemma 93 (High-degree Jacobian ideal membership)**.** *Let K be a field, let $(x_i)_{i\in I}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$ with $x_i \in \mathrm{P}_{w_i}$, and choose principal series b_i of degree w_i representing the x_i. Assume that evaluation at (x_i) is injective in every weighted degree below $\alpha < \omega_1$. For a series a of ordinal value below ω^{α}, write P_a for its resulting polynomial representative.*

Let $F \in K[X_i : i \in I]$ be weighted-homogeneous of degree α, and suppose $w_i < \alpha$ for every variable X_i occurring in F. Choose ordinals α_1, α'' such that $\alpha_1 \leq \alpha''$ and $\alpha_1 \leq \alpha$. Suppose that, for some $\varepsilon_1 > 0$,

$$v_J((F(b_i))^{|\gamma}) < \omega^{\alpha_1} \qquad (-\varepsilon_1 < \gamma < 0).$$

Assume also that every term obtained from a monomial of F by replacing the weights of at least two variable factors, counted with multiplicity, by strictly smaller ordinals has weight below α''.

Fix $i' \in I$ and an ordinal τ with $\alpha'' \le \tau \oplus w_{i'}$. Then, for every $\gamma < 0$ sufficiently close to 0,

$$\big(P_{((\partial_{i'}F)(b_i))^{|\gamma}}\big)_{\ge\tau} \in \big(\partial_j F : j \in \operatorname{vars}(F),\ w_{i'} < w_j\big).$$

Here the subscript $\ge \tau$ denotes the sum of the monomials of weighted degree at least τ.

Proof. Apply Lemma 92 (*Differentiated polynomial Leibniz rule*). In each of the first three terms, adjoining one factor of weight $w_{i'}$ to a monomial gives weight below α''. The hypothesis $\alpha'' \le \tau \oplus w_{i'}$ therefore makes every such monomial have weight below τ. For the derivative of $P_{(F(b_i))^{|\gamma}}$, the defining degree bound for the representative makes its monomial weights less than $\alpha_1 \le \alpha''$; for the two remainder terms this is exactly the assumed bound on terms with at least two translated factors. Their components of degree at least τ consequently vanish.

It remains to consider

$$\sum_{j\in\operatorname{vars}(F)} (\partial_{i'} P_{b_j^{|\gamma}})\, \partial_j F.$$

By Lemma 82 (*Ordinal-value bound for proper translated truncations of a principal series*), the first factor has weighted degree below w_j. It is therefore zero when $w_j \le w_{i'}$. The remaining summands are multiples of the displayed generators. Each $\partial_j F$ is weighted-homogeneous when $w_j \preccurlyeq \alpha$ in the algebraic order, and otherwise vanishes by Lemma 12 (*Vanishing criterion for partial derivatives of weighted-homogeneous polynomials*). Hence taking the components of degree at least τ preserves membership in their ideal. □

`Berarducci.Lifts.exists_forall_componentsGE_pol_translatedTruncation_aeval_pderiv_mem`

Lemma 94 (Ideal membership in $\mathrm{P}_{\tau+1}$ from translated truncations)**.** *Let K be a field, let $(x_i)_{i\in I}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$, with $x_i \in \mathrm{P}_{w_i}$, and choose series b_i representing x_i. Fix $\alpha < \omega_1$, and assume that evaluation at (x_i) is injective in every weighted degree below α.*

Let $Q_1, \dots, Q_m \in K[X_i : i \in I]$ be weighted homogeneous of degrees $\sigma_1, \dots, \sigma_m$, each a limit ordinal or 0. Let $\tau + 1 < \alpha$, and let $u \in K((\mathbb{R}^{\le 0}))$ satisfy $v_J(u) < \omega^{\tau+2}$. Suppose there is $\varepsilon > 0$ such that, for every $-\varepsilon < \gamma < 0$,

$$v_J(u^{|\gamma}) < \omega^{\tau+1}, \qquad \operatorname{pol}(u^{|\gamma})_{\ge\tau} \in (Q_1, \dots, Q_m).$$

Then the class $u + J_{\omega^{\tau+1}} \in \mathrm{P}_{\tau+1}$ belongs, inside $\widehat{\mathrm{P}}$, to the ideal generated by $Q_1(x_i), \dots, Q_m(x_i)$.

Proof. For every $\gamma \in (-\varepsilon, 0)$, Lemma 83 (*Homogeneous component representing a class in* P_β) identifies the class of $u^{|\gamma}$ in P_τ with the evaluation of the degree-τ component of $\operatorname{pol}(u^{|\gamma})$. Since the part of degree at least τ lies in the homogeneous ideal $(Q_1, \dots, Q_m)$, its degree-τ component does too. Hence every value of the derivative of $u + J_{\omega^{\tau+1}}$, sufficiently close to 0, lies in the ideal generated by the $Q_j(x_i)$. Applying Proposition 88 (*Derivative criterion for homogeneous ideal membership*) gives the asserted ideal membership in $\widehat{\mathrm{P}}$. □

`Berarducci.Lifts.of_principalComponentMk_mem_span_of_forall_componentsGE_mem`

6 Limit ordinals in the degree induction

At a successor degree, a smaller term belongs to a specific earlier stage of the induction. A limit ordinal degree has no immediately previous stage. Translated truncations may approach the target degree without all fitting below one smaller bound. This is the gap left by the preceding chapter.

The proof cuts supports into intervals and moves the cutoff so that translated truncations realise suitable smaller ordinal values. The earlier lemmas on Cantor normal form and Hessenberg's natural sum keep track of the high and low parts of the weights as the cutoff moves. They establish two simultaneous requirements: a variable of maximal weight occurs linearly, and each required partial derivative is a finite linear combination of partial derivatives indexed by variables whose low-degree part either equals the full part below the cutoff or is proper and does not precede the distinguished least term. Only in the latter case must the weight be strictly larger. Differentiating one of these identities in the maximal variable makes its right-hand side vanish while its left-hand side remains non-zero. This contradiction supplies the case of a limit ordinal degree in the algebraic-independence test.

Lemma 95 (Linearity when $(w_B)_{<\beta} \oplus \nu \neq \lambda_0$ for every ν)**.** *Let K be a field. Let $(x_i)_{i\in I}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$, with $x_i \in \mathrm{P}_{w_i}$, and choose series b_i representing x_i.*

Let $0 \neq F \in K[X_i : i \in I]$ be weighted homogeneous of degree α with $F(x) = 0$. Suppose every variable of F has weight below α and zero constant Cantor coefficient. Choose X_{B_0} of maximal weight among the variables of F, and put $D = \deg_{X_{B_0}} F$. Suppose there are ordinals $\beta, \Delta, \lambda_0, \alpha_1$ such that

$$\Delta \neq 0, \qquad \Delta \oplus D w_{B_0} = \alpha, \qquad \lambda_0 < \alpha_{<\beta},$$

every Cantor term of Δ is at least ω^β, and the last Cantor term of w_{B_0} is below ω^β. Assume also

$$\alpha_1 \leq \alpha_{\geq\beta} \oplus \lambda_0, \qquad \alpha_1 \leq \alpha,$$

that for some $\varepsilon_1 > 0$ every $\gamma \in (-\varepsilon_1, 0)$ satisfies

$$v_J\big(F(b)^{|\gamma}\big) < \omega^{\alpha_1},$$

and that in the convolution expansion of any monomial of F, every term ρ using at least two translated truncations satisfies $\rho < \alpha_{\geq\beta} \oplus \lambda_0$.

Let X_B occur in F, and suppose

$$0 < (w_B)_{<\beta} \neq \alpha_{<\beta}, \qquad (w_B)_{<\beta} \not\preccurlyeq \lambda_0.$$

Then X_B has exponent 1 in every monomial of F in which it occurs.

Proof. Put $t = (w_B)_{<\beta}$. If some monomial contains X_B at least twice, write it as $M' X_B^2$. The finite part of t is zero, so its last Cantor term is ω^e for some $e \neq 0$. Weighted homogeneity gives

$$(\deg M')_{<\beta} \oplus t \oplus t = \alpha_{<\beta}.$$

The two-truncation bound says that for all $\rho_1, \rho_2 < t$,

$$(\deg M')_{<\beta} \oplus \rho_1 \oplus \rho_2 \leq \lambda_0.$$

Since $\lambda_0 < \alpha_{<\beta}$, Lemma 26 (*Hessenberg-sum decomposition with equal last Cantor terms*) gives $t \preccurlyeq \lambda_0$, a contradiction. ∎

`Berarducci.Lifts.LimitOrdinalRelationAtCutoff.apply_eq_one_of_hasProperLowDegreePartNotAlgebraicLE`

Lemma 96 (Separation of last Cantor terms under a two-factor bound)**.** *Let K be a field. Let $(x_i)_{i\in I}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$, with $x_i \in \mathrm{P}_{w_i}$, and choose series b_i representing x_i.*

Let $0 \neq F \in K[X_i : i \in I]$ be weighted homogeneous of degree α with $F(x) = 0$. Suppose every variable of F has weight below α and zero constant Cantor coefficient. Choose X_{B_0} of maximal weight among the variables of F, and put $D = \deg_{X_{B_0}} F$. Suppose there are ordinals $\beta, \Delta, \lambda_0, \alpha_1$ such that

$$\Delta \neq 0, \qquad \Delta \oplus D w_{B_0} = \alpha, \qquad \lambda_0 < \alpha_{<\beta},$$

every Cantor term of Δ is at least ω^β, and the last Cantor term of w_{B_0} is below ω^β. Assume also

$$\alpha_1 \leq \alpha_{\geq\beta} \oplus \lambda_0, \qquad \alpha_1 \leq \alpha,$$

that for some $\varepsilon_1 > 0$ every $\gamma \in (-\varepsilon_1, 0)$ satisfies

$$v_J\big(F(b)^{|\gamma}\big) < \omega^{\alpha_1},$$

and that in the convolution expansion of any monomial of F, every term ρ using at least two translated truncations satisfies $\rho < \alpha_{\geq\beta} \oplus \lambda_0$.

Let distinct variables X_B, X_C occur in the same monomial of F. Suppose

$$0 < (w_B)_{<\beta} \neq \alpha_{<\beta}, \qquad (w_B)_{<\beta} \oplus \nu \neq \lambda_0 \quad \textit{for every ordinal } \nu,$$

and $(w_C)_{<\beta} \neq 0$. If the last Cantor terms of $(w_B)_{<\beta}$ and $(w_C)_{<\beta}$ are respectively ω^{e_B} and ω^{e_C}, then

$$e_B < e_C, \qquad (\lambda_0)_{\geq e_C} = (\alpha_{<\beta})_{\geq e_C}.$$

Proof. Write the chosen monomial as $M' X_B X_C$ and put $O = (\deg M')_{<\beta}$. Weighted homogeneity gives

$$O \oplus (w_B)_{<\beta} \oplus (w_C)_{<\beta} = \alpha_{<\beta},$$

while the convolution-remainder hypothesis bounds $O \oplus \rho_B \oplus \rho_C$ by λ_0 whenever $\rho_B < (w_B)_{<\beta}$ and $\rho_C < (w_C)_{<\beta}$.

The exponent e_B is nonzero because the variables of F have zero constant Cantor coefficient. If $e_B = e_C$, then Lemma 26 (*Hessenberg-sum decomposition with equal last Cantor terms*) gives an ordinal ν with $(w_B)_{<\beta} \oplus \nu = \lambda_0$. If $e_C < e_B$, the same conclusion follows from Lemma 25 (*Hessenberg-sum decomposition with unequal last Cantor terms*). Both contradict the hypothesis, so $e_B < e_C$. The upper-Cantor-term rigidity used in the proof of the unequal-last-term case, applied to the same bound, then gives $(\lambda_0)_{\geq e_C} = (\alpha_{<\beta})_{\geq e_C}$. ☐

`Berarducci.Lifts.LimitOrdinalRelationAtCutoff.lt_and_partGE_eq_of_hasProperLowDegreePartNotAlgebraicLE`

Lemma 97 (Upper Cantor terms at the complementary low weight)**.** *Let K be a field. Let $(x_i)_{i\in I}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$, with $x_i \in \mathrm{P}_{w_i}$, and choose series b_i representing x_i.*

Let $0 \neq F \in K[X_i : i \in I]$ be weighted homogeneous of degree α with $F(x) = 0$. Suppose every variable of F has weight below α and zero constant Cantor coefficient. Choose X_{B_0} of maximal *weight among the variables of F, and put $D = \deg_{X_{B_0}} F$. Suppose there are ordinals $\beta, \Delta, \lambda_0, \alpha_1$ such that*

$$\Delta \neq 0, \qquad \Delta \oplus D w_{B_0} = \alpha, \qquad \lambda_0 < \alpha_{<\beta},$$

every Cantor term of Δ is at least ω^β, and the last Cantor term of w_{B_0} is below ω^β. Assume also

$$\alpha_1 \leq \alpha_{\geq\beta} \oplus \lambda_0, \qquad \alpha_1 \leq \alpha,$$

that for some $\varepsilon_1 > 0$ every $\gamma \in (-\varepsilon_1, 0)$ satisfies

$$v_J\big(F(b)^{|\gamma}\big) < \omega^{\alpha_1},$$

and that in the convolution expansion of any monomial of F, every term ρ using at least two translated truncations satisfies $\rho < \alpha_{\geq\beta} \oplus \lambda_0$.

Let X_B occur in F, and suppose

$$0 < (w_B)_{<\beta} \neq \alpha_{<\beta}, \qquad (w_B)_{<\beta} \oplus \nu \neq \lambda_0 \quad \textit{for every ordinal } \nu.$$

If

$$c \oplus (w_B)_{<\beta} = \alpha_{<\beta}$$

and the last Cantor term of c is ω^e, then

$$(\lambda_0)_{\geq e} = (\alpha_{<\beta})_{\geq e}.$$

Proof. Choose a monomial containing X_B. By Lemma 95 (*Linearity when $(w_B)_{<\beta} \oplus \nu \neq \lambda_0$ for every ν*), X_B has exponent one in this monomial. After deleting it, the natural sum of the remaining nonzero parts below β is c. This family is nonempty because otherwise $(w_B)_{<\beta} = \alpha_{<\beta}$.

The last Cantor term of a finite natural sum of nonzero ordinals is the last Cantor term of one of its summands. Choose a remaining variable X_C that supplies ω^e. Then Lemma 96 (*Separation of last Cantor terms under a two-factor bound*) applied to X_B and X_C gives $(\lambda_0)_{\geq e} = (\alpha_{<\beta})_{\geq e}$. ☐

`Berarducci.Lifts.LimitOrdinalRelationAtCutoff.partGE_lam₀_eq_of_hasProperLowDegreePartNotAlgebraicLE`

Lemma 98 (Translated truncations of interval pieces)**.** *Let K be a field, let $u \in K((\mathbb{R}))$, and let $a, b, \xi \in \mathbb{R}$ satisfy $a - b < \xi \leq 0$. Put*

$$p = t^{-b} \sum_{a < x \leq b} u_x t^x.$$

Then

$$p^{|\xi} \equiv u^{|b+\xi} \pmod{J}.$$

Proof. The number $a - b - \xi$ is negative. For every exponent $\delta > a - b - \xi$, compare coefficients. If $\delta \leq 0$, then $a < b + \xi + \delta \leq b$, and both coefficients are $u_{b+\xi+\delta}$. If $\delta > 0$, both coefficients are zero. The support of the difference is therefore bounded above by $a - b - \xi < 0$, so the difference lies in J. ☐

`Berarducci.translatedTruncation_window_sub_mem`

Lemma 99 (Translated truncations of shifted truncation sums)**.** *Let K be a field, let $w_k \in K((\mathbb{R}^{\leq 0}))$, and let $c_k, \gamma_k \in \mathbb{R}$. Suppose that (γ_k) is strictly increasing, $\gamma_k < 0$, and $\gamma_k \leq \gamma_{k+1} + c_{k+1}$ for every k. Put*

$$s = \sum_k w_{k>c_k} t^{\gamma_k}.$$

For every k and every $c_k < \xi \leq 0$,

$$s^{|\gamma_k+\xi} \equiv w_k^{|\xi} \pmod J.$$

Proof. The shifted support intervals are strictly ordered, so Lemma 18 (*Well-ordering of a countable ordered union*) ensures that their union is well ordered and the series s is defined. Fix k and $c_k < \xi \leq 0$. Since $c_k - \xi < 0$, it suffices to compare coefficients at exponents $\delta > c_k - \xi$. If $\delta \leq 0$, then $\gamma_k + \xi + \delta$ lies in the k-th interval, so the coefficient of s comes from its k-th summand and equals the coefficient of w_k at $\xi + \delta$. If $\delta > 0$, both translated truncations have zero coefficient. Their difference is supported at or below $c_k - \xi < 0$, and therefore lies in J. □

`Berarducci.translatedTruncation_sumAlongCutoffsSeries_sub_mem`

Lemma 100 (Order type of a sum along cutoffs)**.** *Let K be a field, let $w_k \in K((\mathbb{R}^{\leq 0}))$, let (γ_k) be a strictly increasing sequence of negative real numbers, and let $c_k \in \mathbb{R}$ satisfy $\gamma_k \leq \gamma_{k+1} + c_{k+1}$ for every k. Put*

$$s = \sum_k (w_k)_{>c_k} t^{\gamma_k}.$$

If $\mathrm{ot}(\mathrm{supp}(w_k)) < \omega^\rho$ *for every k, then* $\mathrm{ot}(\mathrm{supp}(s)) \leq \omega^\rho$.

Proof. The k-th summand is supported in $(\gamma_k + c_k, \gamma_k]$, and its support order type is at most $\mathrm{ot}(\mathrm{supp}(w_k))$. The cutoff inequality strictly orders these supports by k. By Lemma 20 (*Countable ordered unions below ω^e*), their union has order type at most ω^ρ. The support of s is contained in this union. □

`Berarducci.supportOrderType_sumAlongCutoffsSeries_le`

Lemma 101 (Finiteness of translated truncations above an ordinal-value bound)**.** *Let K be a field, let $d \in K((\mathbb{R}^{\leq 0}))$, and let $\rho < \omega_1$. If*

$$\mathrm{ot}(\mathrm{supp}\, d) < \omega^{\rho+1},$$

then the set

$$\{\xi \leq 0 : v_J(d^{|\xi}) \geq \omega^\rho\}$$

is finite.

Proof. Among negative cutoffs, infinitely many such points would contain a strictly increasing sequence. The support in every interval between consecutive points would have order type at least ω^ρ, so these disjoint successive blocks would force the support at or below 0 to have order type at least $\omega^\rho \cdot \omega = \omega^{\rho+1}$, a contradiction. Adding the possible cutoff 0 preserves finiteness. □

`Berarducci.finite_setOf_wpow_le_ordinalValue_translatedTruncation`

Lemma 102 (Support-order reduction at a successor degree)**.** *Let K be a field, let $(x_i)_{i\in I}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$, with $x_i \in \mathrm{P}_{w_i}$, and choose principal series b_i representing the x_i. Fix $\alpha < \omega_1$, and assume that evaluation at (x_i) is injective in every weighted degree below α.*

Let $Q_1, \dots, Q_m \in K[X_i : i \in I]$ be weighted homogeneous of degrees $\sigma_1, \dots, \sigma_m$, each a limit ordinal or 0*. Let $\rho_1, \dots, \rho_m$ and τ satisfy*

$$\rho_j \oplus \sigma_j = \tau + 1, \qquad \rho_j \oplus \theta < \tau \quad (\theta < \sigma_j), \qquad \tau + 1 < \alpha.$$

Let $d \in K((\mathbb{R}^{\le 0}))$ satisfy $\mathrm{ot}(\operatorname{supp} d) < \omega^{\tau+2}$*, and suppose that, for every $\xi \le 0$,*

$$\mathrm{pol}(d^{|\xi})_{\ge\tau} \in (Q_1, \dots, Q_m).$$

Then there are $u_1, \dots, u_m \in K((\mathbb{R}^{\le 0}))$ such that $\mathrm{ot}(\operatorname{supp} u_j) < \omega^{\rho_j+1}$ *for every j and*

$$\mathrm{ot}\left(\operatorname{supp}\left(d - \sum_j u_j Q_j(b_i)\right)\right) < \omega^{\tau+1}.$$

Proof. By Lemma 101 (*Finiteness of translated truncations above an ordinal-value bound*), the set L of cutoffs $\xi \le 0$ for which $v_J(d^{|\xi}) \ge \omega^{\tau+1}$ is finite. For each $\xi \in L$, choose an interval immediately below ξ containing no other point of L. Applying Lemma 94 (*Ideal membership in $\mathrm{P}_{\tau+1}$ from translated truncations*) to $d^{|\xi}$ places its class in $\mathrm{P}_{\tau+1}$ in the ideal generated by the $Q_j(x_i)$. By Lemma 3 (*Homogeneous decomposition in a finitely generated graded ideal*), there are cofactor classes of degrees ρ_j; choose principal representatives $w_{\xi j}$, shift them to ξ, and sum over the finite set L.

At a cutoff in L, the term placed there cancels its class. At every other placement, the translated truncation either lies in J, because its cutoff is positive, or is a proper translated truncation bounded by Lemma 85 (*Translated truncations of weighted-homogeneous polynomials*) and the separation hypothesis. Thus every translated truncation of the remainder has ordinal value below $\omega^{\tau+1}$. By Corollary 81 (*Support-order bound from uniformly small translated truncations*), its support has order type below $\omega^{\tau+1}$. Each u_j is a finite sum of shifts of series with support order type at most ω^{ρ_j}, so its support has order type below ω^{ρ_j+1}. □

`Berarducci.Lifts.IsPrincipal.exists_supportOrderType_sub_sum_mul_aeval_lt`

Lemma 103 (Realisation of lower ordinal values by translated truncations)**.** *Let K be a field and let $u \in K((\mathbb{R}^{\le 0}))$. If*

$$v_J(u) = \omega^\beta \qquad \text{and} \qquad \rho < \beta,$$

then, for every $\theta < 0$, there is a γ such that

$$\theta < \gamma < 0 \qquad \text{and} \qquad v_J(u^{|\gamma}) = \omega^\rho.$$

Proof. Since $\rho < \beta$, the ordinal value of u is greater than 1. By Fact 31 (*Support-tail characterization of the ordinal value (Ber00, Definition 5.2)*), choose $\eta > \theta$ such that the negative support tail

$$B = \operatorname{supp}(u) \cap (\eta, 0)$$

has order type ω^β. Let S be the initial segment of B of order type ω^ρ, and put $\gamma = \sup S$. Then $\theta < \gamma < 0$.

If $\rho = 0$, the set S is a singleton. Thus γ is a support point with a gap immediately below it. The translated truncation $u^{|\gamma}$ has a nonzero constant term and ordinal value at most 1, hence ordinal value exactly $1 = \omega^0$.

Suppose $\rho > 0$. Then ω^ρ is an additively principal limit ordinal, so S has no largest element. The set S is a final segment of the support strictly below γ, which gives $v_J(u^{|\gamma}) \le \omega^\rho$. Conversely, every interval immediately below γ contains a nonempty final segment of S, and every such final segment still has order type ω^ρ. The support-tail characterisation of v_J therefore gives the reverse inequality. ☐

`Berarducci.exists_ordinalValue_translatedTruncation_eq_wpow_of_lt`

Lemma 104 (Maximal-variable linearity for the ordinal-value degree)**.** *Let K be a field of characteristic zero. Let $(x_i)_{i\in I}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$, with $x_i \in \mathrm{P}_{w_i}$, and choose principal series b_i of degree w_i representing x_i. Assume evaluation at (x_i) is injective on every homogeneous degree below α.*

Let $0 \ne F \in K[X_i : i \in I]$ be weighted homogeneous of degree α with $F(x) = 0$. Suppose every variable of F has weight below α and zero constant Cantor coefficient. Choose X_{B_0} of maximal weight among the variables of F, and put $D = \deg_{X_{B_0}} F$. Suppose there are ordinals $\beta, \Delta, \lambda_0, \alpha_1$ such that

$$\Delta \ne 0, \qquad \Delta \oplus D w_{B_0} = \alpha, \qquad \lambda_0 < \alpha_{<\beta},$$

every Cantor term of Δ is at least ω^β, and the last Cantor term of w_{B_0} is below ω^β. Assume also

$$\alpha_1 \le \alpha_{\ge\beta} \oplus \lambda_0, \qquad \alpha_1 \le \alpha,$$

that for some $\varepsilon_1 > 0$ every $\gamma \in (-\varepsilon_1, 0)$ satisfies

$$v_J\big(F(b)^{|\gamma}\big) < \omega^{\alpha_1},$$

and that in the convolution expansion of any monomial of F, every term ρ using at least two translated truncations satisfies $\rho < \alpha_{\ge\beta} \oplus \lambda_0$.

Then $D = 1$.

Proof. Suppose $D \ge 2$, put $t = (w_{B_0})_{<\beta}$, and set $\Theta = \partial_{B_0} F$. The hypotheses give $\alpha_{<\beta} = Dt$, while t is a nonzero limit ordinal. The polynomial Θ is nonzero and homogeneous of degree

$$h = \Delta \oplus (D-1) w_{B_0} < \alpha.$$

Since evaluation is injective in degree h, the homogeneous evaluation $\Theta(b)$ has ordinal value ω^h.

Since $\lambda_0 \oplus 1 < (D-1)t \oplus t$, Lemma 21 (*Cofinality below a Hessenberg sum*) gives $s < (D-1)t$ with $\lambda_0 \oplus 1 \le s \oplus t$. Put $\tau = h_{\ge\beta} \oplus s$. Then $\tau < h$ and $\alpha_{\ge\beta} \oplus \lambda_0 \le \tau \oplus w_{B_0}$.

By Lemma 93 (*High-degree Jacobian ideal membership*), the components of weight at least τ in a polynomial representing a sufficiently late translated truncation of $\Theta(b)$ lie in the ideal generated by $\partial_B F$ for variables with $w_B > w_{B_0}$. Maximality of B_0 makes this ideal zero. On the other hand, Lemma 103 (*Realisation of lower ordinal values by translated truncations*) supplies such a translated truncation with ordinal value exactly ω^τ. This ordinal value is nonzero, and $\tau < h < \alpha$, so its unique polynomial representative with monomial weights below α is defined. By Proposition 75 (*Ordinal value of a polynomial representative*), its largest monomial weight is τ; hence its component of weight τ is nonzero, a contradiction. ☐

`Berarducci.Lifts.LimitOrdinalRelationAtCutoff.degreeOf_eq_one`

Lemma 105 (Support-order reduction by interval decomposition)**.** *Let K be a field and let $\mathcal{B}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$ with principal representatives b_B. Assume evaluation is injective below α. Let I be a finite index set, and let $Q_j \in K[X_B : B \in \mathcal{B}]$ be weighted homogeneous of degree σ_j, for $j \in I$. Let $\tau + 1 < \mu < \alpha$, and suppose*

$$\rho_j \oplus \sigma_j = \mu, \qquad \rho_j \oplus \theta < \tau \quad (\theta < \sigma_j)$$

for every j.

Let $u \in K((\mathbb{R}^{\leq 0}))$ satisfy

$$\mathrm{ot}(\mathrm{supp}(u)) < \omega^{\mu+1}$$

and, for every $\xi \leq 0$,

$$\mathrm{pol}_{<\alpha}(u^{|\xi})_{\geq\tau} \in (Q_j : j \in I).$$

Suppose every series c with $\mathrm{ot}(\mathrm{supp}(c)) < \omega^{\mu}$ and the same high-degree ideal condition admits series w_j such that

$$\mathrm{ot}(\mathrm{supp}(w_j)) < \omega^{\rho_j}$$

and

$$v_J\left((c - \sum_j w_j Q_j(b_B))^{|\xi}\right) < \omega^{\tau+1} \quad (\xi \leq 0).$$

Then there are series u_j such that

$$\mathrm{ot}(\mathrm{supp}(u_j)) < \omega^{\rho_j+1},$$

$$\mathrm{ot}\left(\mathrm{supp}(u - \sum_j u_j Q_j(b_B))\right) < \omega^{\mu},$$

and every translated truncation of the remainder satisfies the same high-degree ideal condition.

Proof. By Lemma 101 (*Finiteness of translated truncations above an ordinal-value bound*), only finitely many cutoffs $\xi \leq 0$ have $v_J(u^{|\xi}) \geq \omega^{\mu}$. Around each such cutoff choose an interval containing no other large cutoff. Lemma 98 (*Translated truncations of interval pieces*) transfers the high-degree ideal condition to the corresponding interval piece, and Corollary 81 (*Support-order bound from uniformly small translated truncations*) gives support order below ω^{μ} for its successive subpieces. Apply the assumed induction hypothesis to those subpieces.

Assemble their cofactors as sums along cutoffs. Lemma 99 (*Translated truncations of shifted truncation sums*) identifies their translated truncations, while Lemma 100 (*Order type of a sum along cutoffs*) bounds each local combined cofactor by ω^{ρ_j}. This non-strict bound is sufficient: there are only finitely many exceptional cutoffs, so the finite sum of their shifted local corrections has support order strictly below ω^{ρ_j+1}. The contrapositive consequence of Lemma 103 (*Realisation of lower ordinal values by translated truncations*) bounds the local residual value by $\omega^{\tau+2} \leq \omega^{\mu}$, since $\tau + 1 < \mu$.

Shift the local corrections back to their exceptional cutoffs and add them. The corrections preserve the high-degree ideal condition by Lemma 86 (*High-degree components of translated truncations of a polynomial multiple*). They cancel all large cutoffs, so Corollary 81 (*Support-order bound from uniformly small translated truncations*) gives support order below ω^{μ} for the final remainder. ☐

`Berarducci.Lifts.IsPrincipal.exists_supportOrderType_sub_sum_mul_aeval_lt_of_pieces`

Lemma 106 (Translated-truncation approximation at intermediate degrees)**.** *Let K be a field and let $\mathcal{B}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$ with principal representatives b_B. Assume evaluation is injective below α. Let I be a finite index set, and let $Q_j \in K[X_B : B \in \mathcal{B}]$ be weighted homogeneous of nonzero degree σ_j that is a limit ordinal. Let $\mu < \alpha$, and suppose*

$$\rho_j \oplus \sigma_j = \mu, \qquad \rho_j \oplus \theta < \tau \quad (\theta < \sigma_j)$$

for every j.

For every μ' with $\tau < \mu' \leq \mu$, every family $(\rho'_j)_{j \in I}$ satisfying $\rho'_j \oplus \sigma_j = \mu'$, and every series $u \in K((\mathbb{R}^{\leq 0}))$ such that

$$\mathrm{ot}(\mathrm{supp}(u)) < \omega^{\mu'+1}$$

and

$$\mathrm{pol}_{<\alpha}(u^{|\xi})_{\geq \tau} \in (Q_j : j \in I) \quad (\xi \leq 0),$$

there are series u_j with

$$\mathrm{ot}(\mathrm{supp}(u_j)) < \omega^{\rho'_j+1}$$

and

$$v_J\left((u - \sum_j u_j Q_j(b_B))^{|\xi}\right) < \omega^{\tau+1} \quad (\xi \leq 0).$$

Proof. Use well-founded induction on μ'. By Lemma 28 (*Intermediate ordinals below a Hessenberg sum*), each $\rho'_j \leq \rho_j$, so the separation inequalities remain valid; the same result supplies cofactor degrees at every smaller degree used below.

For a series with support order below $\omega^{\mu'}$, choose $\mu'' < \mu'$ such that its support order is below $\omega^{\mu''+1}$. If $\mu'' \leq \tau$, zero cofactors suffice. Otherwise apply the induction hypothesis at μ''.

If $\mu' = \tau + 1$, apply Lemma 102 (*Support-order reduction at a successor degree*), then convert its support bound into the required translated-truncation bound. If $\tau + 1 < \mu'$, apply Lemma 105 (*Support-order reduction by interval decomposition*) to obtain first cofactors whose remainder has support order below $\omega^{\mu'}$ and retains the high-degree ideal condition. Apply the auxiliary induction hypothesis to this remainder and add the two cofactor families. The natural-sum support estimate gives the required bounds. ☐

`Berarducci.Lifts.IsPrincipal.exists_forall_ordinalValue_translatedTruncation_sub_sum_mul_aeval_lt`

Proposition 107 (Ideal membership from translated truncations)**.** *Let K be a field and let $\mathcal{B}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$ with principal representatives b_B. Assume evaluation is injective below α. Let I be a finite index set, and let $Q_j \in K[X_B : B \in \mathcal{B}]$ be weighted homogeneous of nonzero degree σ_j that is a limit ordinal. Suppose*

$$\rho_j \oplus \sigma_j = \mu < \alpha, \qquad \tau + 1 < \mu, \qquad \rho_j \oplus \theta < \tau \quad (\theta < \sigma_j)$$

for every j.

Let $u \in K((\mathbb{R}^{\leq 0}))$ have ordinal value $v_J(u) = \omega^\mu$. Suppose there is $\eta < 0$ such that

$$\mathrm{pol}_{<\alpha}(u^{|\gamma})_{\geq \tau} \in (Q_j : j \in I)$$

for every $\eta < \gamma < 0$. Then the degree-μ class $u + J_{\omega^\mu} \in \mathrm{P}_\mu$, embedded in $\widehat{\mathrm{P}}$, lies in

$$(Q_j(\mathcal{B}) : j \in I) \subseteq \widehat{\mathrm{P}}.$$

Proof. Choose a negative interval on which every support tail of u has order type $v_J(u) = \omega^\mu$, and cut that tail along a sequence $\gamma_k \uparrow 0$. Each interval piece has support order strictly below ω^μ and inherits the high-degree ideal condition. For a piece whose support degree is at most τ, take zero cofactors; otherwise apply Lemma 106 (*Translated-truncation approximation at intermediate degrees*) at that degree.

Combine the piecewise cofactors into series C_j. Their support bounds give $v_J(C_j) < \omega^{\rho_j+1}$, while every translated truncation sufficiently close to zero of $u - \sum_j C_j Q_j(b_B)$ has ordinal value below $\omega^{\tau+1}$. Consequently the residual series has ordinal value below $\omega^{\tau+2}$. Since $\tau+1 < \mu$, one has $\tau + 2 \le \mu$, and hence this value is at most ω^μ. Thus u and $\sum_j C_j Q_j(b_B)$ define the same degree-μ class. Multiplicativity of representatives then places this class in $(Q_j(\mathcal{B}) : j \in I)$. ☐

`Berarducci.Lifts.IsPrincipal.of_principalComponentMk_mem_span_of_forall_componentsGE_mem`

Proposition 108 (Partial-derivative syzygy when $(w_{B'})_{<\beta} \oplus \eta = \lambda_0$ for some η)**.** *Let K be a field. Let $(x_i)_{i\in I}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$, with $x_i \in \mathrm{P}_{w_i}$, and choose principal series b_i of degree w_i representing x_i. Assume evaluation at (x_i) is injective on every homogeneous degree below α.*

Let $0 \neq F \in K[X_i : i \in I]$ be weighted homogeneous of degree α with $F(x) = 0$. Suppose every variable of F has weight below α and zero constant Cantor coefficient. Choose X_{B_0} of maximal weight among the variables of F, and put $D = \deg_{X_{B_0}} F$. Suppose there are ordinals $\beta, \Delta, \lambda_0, \alpha_1$ such that

$$\Delta \neq 0, \qquad \Delta \oplus D w_{B_0} = \alpha, \qquad \lambda_0 < \alpha_{<\beta},$$

every Cantor term of Δ is at least ω^β, and the last Cantor term of w_{B_0} is below ω^β. Assume also

$$\alpha_1 \le \alpha_{\ge\beta} \oplus \lambda_0, \qquad \alpha_1 \le \alpha,$$

that for some $\varepsilon_1 > 0$ every $\gamma \in (-\varepsilon_1, 0)$ satisfies

$$v_J\big(F(b)^{|\gamma}\big) < \omega^{\alpha_1},$$

and that in the convolution expansion of any monomial of F, every term ρ using at least two translated truncations satisfies $\rho < \alpha_{\ge\beta} \oplus \lambda_0$.

Let $X_{B'}$ occur in F, and suppose $(w_{B'})_{<\beta} \preccurlyeq \lambda_0$ in the algebraic order. Then there are a finite set E of variables and polynomials $(U_B)_{B\in E}$ such that every $B \in E$ occurs in F and either

$$(w_B)_{<\beta} = \alpha_{<\beta},$$

or

$$0 < (w_B)_{<\beta} \neq \alpha_{<\beta}, \qquad (w_B)_{<\beta} \not\preccurlyeq \lambda_0, \qquad w_{B'} < w_B.$$

Moreover $\partial_{B_0} U_B = 0$ for every $B \in E$, and

$$\partial_{B'} F = \sum_{B\in E} (\partial_B F) U_B.$$

Proof. Use strong induction on the number of variables of F whose weight exceeds $w_{B'}$. The assertion is immediate when $\partial_{B'} F = 0$. Otherwise write

$$h \oplus w_{B'} = \alpha, \qquad (w_{B'})_{<\beta} \oplus \lambda' = \lambda_0, \qquad \tau = h_{\ge\beta} \oplus \lambda'.$$

Weighted homogeneity and injectivity below α give $v_J((\partial_{B'} F)(b)) = \omega^h$ and the required polynomial representatives of its translated truncations. By Lemma 93 (*High-degree Jacobian ideal* *membership*), their components of weight at least τ lie in the ideal generated by $\partial_B F$ with $w_{B'} < w_B$. Apply the induction hypothesis to every such B whose part below β precedes λ_0 in the algebraic order. The remaining generators are exactly those listed in the statement.

The Cantor decompositions give $\tau \oplus 1 < h$. For each remaining generator, Lemma 97 (*Upper Cantor terms at the complementary low weight*) identifies the high Cantor terms of the complementary degree, and Lemma 17 (*Separation below the last Cantor term*) gives $\rho_B \oplus \theta < \tau$ for every $\theta < \sigma_B$, where σ_B is the weight of $\partial_B F$. Thus Proposition 107 (*Ideal membership from translated truncations*), applied to $(\partial_{B'} F)(b)$, puts its degree-h class in the ideal generated by the degree classes of the remaining $\partial_B F$.

Take homogeneous cofactors of the complementary weights in this ideal expression and represent them by weighted-homogeneous polynomials U_B. Evaluating the difference gives zero, and injectivity in degree $h < \alpha$ gives the displayed polynomial identity. Finally each U_B has weight below w_{B_0}, so $\partial_{B_0} U_B = 0$. □

`Berarducci.Lifts.LimitOrdinalRelationAtCutoff.exists_finset_pderiv_eq_sum_of_lowDegreePartAlgebraicLE`

7 Principal RV-elements

The algebraic-independence test is now ready, including its successor and limit cases. We must apply it to principal RV-elements, the objects that record principal series up to terms of lower degree. They form the associated graded algebra for the degree derived from ordinal value. The Cantor–Bendixson construction gives another associated graded algebra, this time described directly through supports. A canonical isomorphism matches the two algebras degree by degree.

Every nonzero homogeneous class has a principal series representative of the exact degree, and every proper translated truncation of that representative has smaller degree. We can therefore carry the Jacobian relation from the previous chapters into the Cantor–Bendixson grading and apply the general test. It follows that every minimal homogeneous generating system of principal RV-elements is algebraically independent. The next chapter combines this absence of relations with generation.

Lemma 109 (Cantor–Bendixson grading of $\widehat{\mathrm{P}}$)**.** *Let $\widehat{\mathrm{P}}$ be the subring of principal elements of $\widehat{\mathrm{RV}}$, equivalently the associated graded K-algebra for $\deg_J$. Define*

$$\delta_{\mathrm{CB}}(b) = \begin{cases} -\infty, & 0 \notin \mathrm{cl}(\mathrm{supp}(b)), \\ \mathrm{rk}_{\mathrm{CB},\mathrm{cl}(\mathrm{supp}(b))}(0), & 0 \in \mathrm{cl}(\mathrm{supp}(b)). \end{cases}$$

There is a canonical isomorphism of K-algebras

$$\widehat{\mathrm{P}} \simeq_K \mathrm{gr}_{\delta_{\mathrm{CB}}} K((\mathbb{R}^{\leq 0})).$$

Proof. The subring $\widehat{\mathrm{P}}$ is the associated graded algebra for $\deg_J$. Lemma 47 (*Cantor–Bendixson formula for* $\deg_J$) identifies $\deg_J$ with δ_{CB}, so the identity on series induces a ring isomorphism between the associated graded algebras. Both scalar embeddings send $k \in K$ to the initial form of the constant series k; hence this is an isomorphism of K-algebras. □

`Berarducci.principalSubringCantorBendixsonAlgEquiv`

Lemma 110 (Principal series representatives for the Cantor–Bendixson grading)**.** *Let K be a field of characteristic zero. For each $i \in I$, let $x_i \in \mathrm{P}_{w_i} \subseteq \widehat{\mathrm{P}}$ and choose a principal series b_i of degree w_i representing x_i. Let δ_{CB} be the Cantor–Bendixson degree identified with* $\deg_J$ *by Lemma 47 (*Cantor–Bendixson formula for $\deg_J$*), and identify $\widehat{\mathrm{P}}$ with* $\mathrm{gr}_{\delta_{\mathrm{CB}}} K((\mathbb{R}^{\leq 0}))$ *by Lemma 109 (*Cantor–Bendixson grading of $\widehat{\mathrm{P}}$*).*

The same series b_i represent the images of the x_i under this isomorphism, and, for every $i \in I$,

$$\delta_{\mathrm{CB}}(b_i) = w_i, \qquad \delta_{\mathrm{CB}}(b_i^{|\gamma}) < w_i \quad \textit{for every } \gamma < 0.$$

Proof. By Lemma 109 (*Cantor–Bendixson grading of* $\widehat{\mathrm{P}}$), the isomorphism between $\widehat{\mathrm{P}}$ and the Cantor–Bendixson associated graded ring is induced by the identity on representing series. Thus the b_i still represent the x_i, and $\delta_{\mathrm{CB}}(b_i) = \deg_J(b_i) = w_i$. For $\gamma < 0$, Lemma 82 (*Ordinal-value bound for proper translated truncations of a principal series*) gives

$$v_J(b_i^{|\gamma}) < \omega^{w_i}.$$

Under the same identification this is precisely $\delta_{\mathrm{CB}}(b_i^{|\gamma}) < w_i$. □

`Berarducci.Lifts.IsPrincipal.cantorBendixson`

Lemma 111 (Transport of the Jacobian syzygy to the Cantor–Bendixson grading)**.** *Let K be a field of characteristic zero. Let $(x_i)_{i \in I}$ be a minimal homogeneous generating system of* $\widehat{\mathrm{P}}$, *with $x_i \in \mathrm{P}_{w_i}$, and choose principal series b_i of degree w_i representing x_i. Under the canonical isomorphism*

$$\widehat{\mathrm{P}} \simeq_K \mathrm{gr}_{\delta_{\mathrm{CB}}} K((\mathbb{R}^{\leq 0})),$$

write $\bar{x}_i$ for the image of x_i, and assume evaluation at $(\bar{x}_i)$ is injective on every homogeneous degree below α.

Let $0 \neq F \in K[X_i : i \in I]$ be weighted homogeneous of degree α with $F(\bar{x}) = 0$. Suppose every variable of F has weight below α and zero constant Cantor coefficient. Choose X_{B_0} of maximal weight among the variables of F, and put $D = \deg_{X_{B_0}} F$. Suppose there are ordinals $\beta, \Delta, \lambda_0, \alpha_1$ such that

$$\Delta \neq 0, \qquad \Delta \oplus D w_{B_0} = \alpha, \qquad \lambda_0 < \alpha_{<\beta},$$

every Cantor term of Δ is at least ω^β, and the last Cantor term of w_{B_0} is below ω^β. Assume also

$$\alpha_1 \leq \alpha_{\geq\beta} \oplus \lambda_0, \qquad \alpha_1 \leq \alpha,$$

that for all $\gamma < 0$ sufficiently close to 0*,*

$$\delta_{\mathrm{CB}}\big(F(b)^{|\gamma}\big) < \alpha_1,$$

and that in the convolution expansion of any monomial of F, every term ρ using at least two translated truncations satisfies $\rho < \alpha_{\geq\beta} \oplus \lambda_0$.

Let $X_{B'}$ occur in F, and suppose $(w_{B'})_{<\beta} \preccurlyeq \lambda_0$ in the algebraic order. Then there are a finite set E of variables and polynomials $(U_B)_{B \in E}$ such that every $B \in E$ occurs in F and either

$$(w_B)_{<\beta} = \alpha_{<\beta},$$

or

$$0 < (w_B)_{<\beta} \neq \alpha_{<\beta}, \qquad (w_B)_{<\beta} \not\preccurlyeq \lambda_0, \qquad w_{B'} < w_B.$$

Moreover $\partial_{B_0} U_B = 0$ for every $B \in E$, and

$$\partial_{B'} F = \sum_{B \in E} (\partial_B F) U_B.$$

Proof. Lemma 109 (*Cantor–Bendixson grading of* $\widehat{\mathrm{P}}$) identifies the Cantor–Bendixson degree with $\deg_J$. It therefore carries $F(\bar{x}) = 0$, injectivity below α, and the eventual degree bound to $\widehat{\mathrm{P}}$, while leaving F, the weights, the cutoff data, and the chosen series unchanged.

Apply Proposition 108 (*Partial-derivative syzygy when* $(w_{B'})_{<\beta} \oplus \eta = \lambda_0$ *for some* η) there. The isomorphism preserves the three conditions on the part of a weight below β, so the same finite set E and the same polynomials U_B give the asserted identity and the equations $\partial_{B_0} U_B = 0$. □

`Berarducci.Lifts.RealPartialDecomposition.lowDegreePartAlgebraicLE_partials`

Theorem 112 (Minimal homogeneous generators of $\widehat{\mathrm{P}}$ are algebraically independent)**.** *Let K be a field of characteristic zero. If $(x_i)_{i \in I}$ is a minimal homogeneous generating system of $\widehat{\mathrm{P}}$, with $x_i \in \mathrm{P}_{w_i}$, then $(x_i)_{i \in I}$ is algebraically independent over K.*

Proof. Choose principal series representatives b_i of the x_i. By Lemma 110 (*Principal series representatives for the Cantor–Bendixson grading*), (x_i) is a minimal homogeneous generating system of $\mathrm{gr}_{\delta_{\mathrm{CB}}} K((\mathbb{R}^{\leq 0}))$, and the same b_i have the required degree and translated-truncation properties for the Cantor–Bendixson grading.

Apply Theorem 63 (*Algebraic independence of a minimal homogeneous generating system*). Its two nontrivial relation hypotheses follow from Lemma 65 (*Maximal-variable linearity for the Cantor–Bendixson degree*) and Lemma 111 (*Transport of the Jacobian syzygy to the Cantor–Bendixson grading*). Thus the transported family is algebraically independent. Injectivity of the graded isomorphism transports algebraic independence back to (x_i) in $\widehat{\mathrm{P}}$. □

`Berarducci.algebraicIndependent_of_isMinimalSystem`

8 Polynomial presentations

Algebraic independence says that the chosen generators satisfy no polynomial relation. To get a polynomial presentation, we must also know that they generate the whole ring. Earlier generation results provide this second half. Scalar extension passes from principal RV-elements to all RV-elements, and lifting the homogeneous generators gives actual series that represent the same variables.

This produces polynomial presentations for the graded algebra, the germ ring, and the full ring of series with non-positive real exponents. The polynomial representatives can also be chosen with a bound on their ordinal value, so the presentation respects the filtration used in the proof. We can now study divisibility and greatest common divisors in polynomial coordinates. The next chapter does so.

Fact 113 (Scalar extension from $\widehat{\mathrm{P}}$ to $\widehat{\mathrm{RV}}$ (LM24, Proposition 6.1.2))**.** *Let K be a field of characteristic zero. Multiplication of initial forms induces an isomorphism of K-algebras*

$$\widehat{\mathrm{P}} \otimes_K K(\mathbb{R}^{\leq 0}) \xrightarrow{\sim} \widehat{\mathrm{RV}}.$$

Proof. Fact 30 (*Principal representatives of* P_α *(LM24, Remark 7.2.4)*) gives in each degree the component equivalence of LM24, Proposition 5.3.1. Fact 34 (*Multiplicativity of the degree (LM24, Theorem D)*) makes multiplication of initial forms respect those components, so their direct sum is an algebra homomorphism. Every element has finite degree support; applying the inverse component maps degree by degree proves bijectivity. $\square$

`Berarducci.principalSubringTensorEquiv`

Theorem 114 (Polynomial presentation of the germ ring over a Cauchy-complete exponent group)**.** *Let K be a field of characteristic zero and let G be a nontrivial, densely ordered abelian group with no least or greatest element and with no least nonzero Archimedean class in the magnitude order. Assume that G is Cauchy complete for its additive uniformity. If J is the ideal of series in $K((G^{\leq 0}))$ whose support is bounded strictly below zero, then for some set I there is a K-algebra isomorphism*

$$K[X_i : i \in I] \simeq K((G^{\leq 0}))/J.$$

Proof. Applying Lemma 2 (*Extension of a degreewise independent family*) to the empty family gives a minimal homogeneous generating system of the associated graded ring of the Cantor–Bendixson degree. Well-order the nonzero Archimedean classes and retain each class that is smaller in magnitude than every earlier class. The resulting family is coinitial in the magnitude order and well-founded when ordered by reverse magnitude. Its groups $G_{\prec\sigma}$ therefore form a decreasing neighbourhood basis at zero. Choose series representing the generators. By Theorem 68 (*Algebraic independence in the Cantor–Bendixson associated graded ring*), these generators are algebraically independent. Minimality makes homogeneous evaluation surjective. Evaluation at the representatives is therefore surjective modulo J, while its highest nonzero weighted-homogeneous component proves injectivity modulo J. $\square$

`HahnSeries.Nonpositive.exists_mvPolynomial_algEquiv_germ`

Theorem 115 (Weighted degree under evaluation at series lifts)**.** *Let K be a field of characteristic zero, and let $(x_i)_{i\in I}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$, with $x_i \in \mathrm{P}_{w_i}$. For each i, choose a series $b_i \in K((\mathbb{R}^{\leq 0}))$ of degree w_i whose initial form is the image of $x_i \otimes 1$ under $\widehat{\mathrm{P}} \otimes_K K(\mathbb{R}^{\leq 0}) \simeq \widehat{\mathrm{RV}}$.*

If $0 \neq F \in K(\mathbb{R}^{\leq 0})[X_i : i \in I]$, then the degree of $F(b_i)$ equals the weighted total degree of F for $\deg(X_i) = w_i$.

Proof. By Fact 113 (*Scalar extension from* $\widehat{\mathrm{P}}$ *to* $\widehat{\mathrm{RV}}$ *(LM24, Proposition 6.1.2)*), the associated graded ring is $\widehat{\mathrm{P}} \otimes_K K(\mathbb{R}^{\leq 0})$. Theorem 112 (*Minimal homogeneous generators of* $\widehat{\mathrm{P}}$ *are algebraically independent*) makes evaluation at $(x_i \otimes 1)$ injective after scalar extension. The hypotheses on b_i therefore make their initial forms an initial-form coordinate system. The top weighted homogeneous component of a non-zero F has non-zero initial form after evaluation, so the degree of $F(b_i)$ is its weighted degree. $\square$

`Berarducci.degree_evalAtLifts_eq`

Theorem 116 (Algebraic independence of series lifts)**.** *Let K be a field of characteristic zero, and let $(x_i)_{i\in I}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$, with $x_i \in \mathrm{P}_{w_i}$. For each*

i, choose a series $b_i \in K((\mathbb{R}^{\leq 0}))$ of degree w_i whose initial form is the image of $x_i \otimes 1$ under $\widehat{\mathrm{P}} \otimes_K K(\mathbb{R}^{\leq 0}) \simeq \widehat{\mathrm{RV}}$. Then evaluation

$$K(\mathbb{R}^{\leq 0})[X_i : i \in I] \longrightarrow K((\mathbb{R}^{\leq 0})), \qquad X_i \longmapsto b_i,$$

is injective.

Proof. Fact 113 (*Scalar extension from $\widehat{\mathrm{P}}$ to $\widehat{\mathrm{RV}}$ (LM24, Proposition 6.1.2)*) identifies the associated graded ring with $\widehat{\mathrm{P}} \otimes_K K(\mathbb{R}^{\leq 0})$. By Theorem 112 (*Minimal homogeneous generators of $\widehat{\mathrm{P}}$ are algebraically independent*), evaluation at the minimal homogeneous generating system is injective, and it remains injective after scalar extension. Hence a non-zero polynomial in the b_i has a non-zero initial form and cannot evaluate to zero. □

`Berarducci.evalAtLifts_injective`

Theorem 117 (Generation of the series ring by series lifts)**.** *Let K be a field of characteristic zero, and let $(x_i)_{i \in I}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$, with $x_i \in \mathrm{P}_{w_i}$. For each i, choose a series $b_i \in K((\mathbb{R}^{\leq 0}))$ of degree w_i whose initial form is the image of $x_i \otimes 1$ under $\widehat{\mathrm{P}} \otimes_K K(\mathbb{R}^{\leq 0}) \simeq \widehat{\mathrm{RV}}$. Then every element of $K((\mathbb{R}^{\leq 0}))$ is the value at (b_i) of a polynomial in $K(\mathbb{R}^{\leq 0})[X_i : i \in I]$.*

Proof. By Fact 113 (*Scalar extension from $\widehat{\mathrm{P}}$ to $\widehat{\mathrm{RV}}$ (LM24, Proposition 6.1.2)*), every element of the associated graded ring comes from a polynomial over $K(\mathbb{R}^{\leq 0})$ in the $x_i \otimes 1$. By Theorem 112 (*Minimal homogeneous generators of $\widehat{\mathrm{P}}$ are algebraically independent*), its weighted homogeneous components are the initial forms of the corresponding values at the b_i. Thus every associated-graded element is a finite sum of initial forms of elements in the range of evaluation. Apply Lemma 16 (*Lifting generation from the associated graded ring*) to that range and the separated degree to conclude that evaluation is surjective. □

`Berarducci.evalAtLifts_surjective`

Theorem 118 (Polynomial presentation of the series ring)**.** *Let K be a field of characteristic zero, and let $(x_i)_{i \in I}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$, with $x_i \in \mathrm{P}_{w_i}$. For each i, choose a series $b_i \in K((\mathbb{R}^{\leq 0}))$ of degree w_i whose initial form is the image of $x_i \otimes 1$ under $\widehat{\mathrm{P}} \otimes_K K(\mathbb{R}^{\leq 0}) \simeq \widehat{\mathrm{RV}}$. Evaluation at the b_i is an isomorphism of $K(\mathbb{R}^{\leq 0})$-algebras*

$$K(\mathbb{R}^{\leq 0})[X_i : i \in I] \simeq K((\mathbb{R}^{\leq 0})),$$

sending X_i to b_i.

Proof. Evaluation is injective by Theorem 116 (*Algebraic independence of series lifts*) and surjective by Theorem 117 (*Generation of the series ring by series lifts*); hence it defines the stated algebra isomorphism. □

`Berarducci.polynomialRingEquiv`

Corollary 119 (Polynomial presentation of $\widehat{\mathrm{P}}$)**.** *Let K be a field of characteristic 0, and let $\mathcal{B} = (B_i)_{i \in I}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$. Evaluation $X_i \mapsto B_i$ is a K-algebra isomorphism $K[X_i : i \in I] \simeq \widehat{\mathrm{P}}$.*

Proof. Principal representatives identify P_0 with K. The homogeneous generation theorem then makes evaluation at $\mathcal{B}$ surjective, while Theorem 112 (*Minimal homogeneous generators of* $\widehat{\mathrm{P}}$ *are algebraically independent*) makes it injective. Hence evaluation is a K-algebra isomorphism. ☐

`Berarducci.algEquivOfIsMinimalSystem`

Corollary 120 (Algebraic independence of homogeneous families in $\widehat{\mathrm{P}}$)**.** *Let K be a field of characteristic* 0*, and let $(x_i)_{i\in I}$ be a family in $\widehat{\mathrm{P}}$ with positive weights w_i and $x_i \in \mathrm{P}_{w_i}$. Suppose that for every β, the images of the indexed subfamily $(x_i)_{w_i=\beta}$ in $\mathrm{P}_\beta/((\widehat{\mathrm{P}}_+)^2 \cap \mathrm{P}_\beta)$ are K-linearly independent. Then evaluation $X_i \mapsto x_i$ is injective; equivalently, (x_i) is algebraically independent over K.*

Proof. Lemma 2 (*Extension of a degreewise independent family*) enlarges the given indexed family, degree by degree, to a minimal homogeneous generating system. Theorem 112 (*Minimal homogeneous generators of* $\widehat{\mathrm{P}}$ *are algebraically independent*) makes this extended system algebraically independent. Algebraic independence passes to the original indexed subfamily. ☐

`Berarducci.aeval_injective_of_independent_mod_decomposableAt`

Corollary 121 (Algebraic independence in additively principal degree)**.** *Let K be a field of characteristic* 0 *and $\gamma < \omega_1$. Every K-linearly independent family $(x_i)_{i\in I}$ in $\mathrm{P}_{\omega^\gamma}$ is algebraically independent over K in $\widehat{\mathrm{P}}$.*

Proof. No two positive degrees have Hessenberg's natural sum ω^γ, so the decomposable subspace in that degree is zero. Linear independence therefore gives independence modulo $(\widehat{\mathrm{P}}_+)^2$. Then Corollary 120 (*Algebraic independence of homogeneous families in* $\widehat{\mathrm{P}}$) gives algebraic independence. ☐

`Berarducci.aeval_injective_of_linearIndependent_of_mem_principalGrading_wpow`

Corollary 122 (Prime elements of additively principal degree)**.** *Let K be a field of characteristic* 0 *and $\gamma < \omega_1$. Every nonzero element of $\mathrm{P}_{\omega^\gamma}$ is prime in $\widehat{\mathrm{P}}$.*

Proof. No two positive degrees have Hessenberg's natural sum ω^γ, so the decomposable subspace in that degree is zero and the nonzero singleton y is independent modulo it. By Lemma 2 (*Extension of a degreewise independent family*), this singleton extends to a minimal homogeneous generating system. By Corollary 119 (*Polynomial presentation of* $\widehat{\mathrm{P}}$), y corresponds to one of the polynomial variables, which is prime; the isomorphism therefore makes y prime in $\widehat{\mathrm{P}}$. ☐

`Berarducci.prime_of_mem_principalGrading_wpow`

Lemma 123 (Surjectivity of evaluation modulo J)**.** *Let K be a field. Let (x_i) be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$, with $x_i \in \mathrm{P}_{w_i}$, and choose $b_i \in K((\mathbb{R}^{\leq 0}))$ representing x_i in degree w_i. Every class in $K((\mathbb{R}^{\leq 0}))/J$ is the class of $F(b_i)$ for some polynomial $F \in K[X_i : i \in I]$.*

Proof. For a representative u of the class, choose $\alpha > v_J(u)$. By Proposition 72 (*Existence of polynomial representatives modulo J*), there is a polynomial F of weighted degree below α such that $F(b_i) \equiv u \pmod{J}$. ☐

Berarducci.Lifts.ordinalValueQuotientAlgHom_surjective

Lemma 124 (Injectivity of evaluation modulo J)**.** *Let K be a field of characteristic zero. Let (x_i) be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$, with $x_i \in \mathrm{P}_{w_i}$, and choose $b_i \in K((\mathbb{R}^{\leq 0}))$ representing x_i in degree w_i. If the class of $F(b_i)$ is zero in $K((\mathbb{R}^{\leq 0}))/J$, then $F = 0$.*

Proof. By Lemma 110 (*Principal series representatives for the Cantor–Bendixson grading*), the chosen representatives give a minimal homogeneous generating system for the Cantor–Bendixson associated graded ring. The two relation hypotheses of Theorem 63 (*Algebraic independence of a minimal homogeneous generating system*) are supplied by Lemma 65 (*Maximal-variable linearity for the Cantor–Bendixson degree*) and Lemma 111 (*Transport of the Jacobian syzygy to the Cantor–Bendixson grading*). Thus evaluation at (x_i) is injective. If $F \neq 0$, its top homogeneous component therefore evaluates nontrivially, so the ordinal-value calculation gives $v_J(F(b_i)) = \omega^{\deg_w(F)} \neq 0$. Hence $F(b_i) \notin J$, contrary to the vanishing of its class. ☐

Berarducci.Lifts.ordinalValueQuotientAlgHom_injective

Theorem 125 (Polynomial presentation of the quotient by J)**.** *Let K be a field of characteristic 0. Let $(x_i)_{i\in\iota}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$, with $x_i \in \mathrm{P}_{w_i}$, and choose $b_i \in K((\mathbb{R}^{\leq 0}))$ representing x_i in degree w_i. There is a K-algebra isomorphism*

$$K((\mathbb{R}^{\leq 0}))/J \longrightarrow K[X_i : i \in \iota]$$

whose inverse sends F to the class of $F(b_i)$ modulo J.

Proof. Evaluation followed by passage to the quotient is surjective by Lemma 123 (*Surjectivity of evaluation modulo J*) and injective by Lemma 124 (*Injectivity of evaluation modulo J*). Its inverse is the stated K-algebra isomorphism. ☐

Berarducci.Lifts.ordinalValueQuotientAlgEquiv

Theorem 126 (Polynomial representative of a germ)**.** *Let K be a field of characteristic 0. Let $(x_i)_{i\in\iota}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$, with $x_i \in \mathrm{P}_{w_i}$, and choose $b_i \in K((\mathbb{R}^{\leq 0}))$ representing x_i in degree w_i. For every $\alpha < \omega_1$ and every $u \in K((\mathbb{R}^{\leq 0}))$ with $v_J(u) < \omega^\alpha$, the polynomial presentation of the quotient by J sends the class of u to its polynomial representative $\mathrm{pol}_\alpha(u)$.*

Proof. By definition, $\mathrm{pol}_\alpha(u)$ has value congruent to u modulo J. The inverse description in Theorem 125 (*Polynomial presentation of the quotient by J*) therefore sends this polynomial to the class of u; applying the isomorphism gives the stated formula. ☐

Berarducci.Lifts.ordinalValueQuotientAlgEquiv_toGerm

Theorem 127 (Polynomial representatives below an ordinal-value bound)**.** *Let K be a field. Let $(x_i)_{i\in\iota}$ be a minimal homogeneous generating system of $\widehat{\mathrm{P}}$, with $x_i \in \mathrm{P}_{w_i}$, and choose $b_i \in K((\mathbb{R}^{\leq 0}))$ representing x_i in degree w_i. For every $\alpha < \omega_1$,*

$$\{u + J : v_J(u) < \omega^\alpha\} = \{F(b_i) + J : \textit{every monomial of } F \textit{ has weighted degree} < \alpha\}.$$

Proof. The forward inclusion follows from Proposition 72 (*Existence of polynomial representatives modulo J*). Conversely, if every monomial of F has weighted degree below α, then $v_J(F(b_i)) < \omega^\alpha$, so the class of $F(b_i)$ belongs to the left-hand set. □

`Berarducci.Lifts.toGerm_image_ordinalValue_lt_eq`

9 Primality and factorisation for real exponents

The polynomial presentations reduce the real-exponent case to familiar commutative algebra. The finite-support coefficient ring has greatest common divisors, and so does the polynomial algebra over it. Carrying this structure back through the presentation shows that the ring of series with non-positive real exponents is a GCD domain. It follows that every series is primal, or equivalently that the ring is pre-Schreier. In particular, every irreducible series is prime.

The chapter also turns several irreducibility criteria into results about prime series, and gives a unique-factorisation theorem under specific conditions on the support order type. For the main proof, the key result is the primality theorem for all series with real exponents. The next chapter uses it at one Archimedean class at a time under assumption $(A1)_\sigma$, which identifies the chosen complement H_σ with the ordered additive group of real numbers.

Fact 128 (Irreducibility from translated truncation dimension)**.** *Let K be a field of characteristic 0 and $a \in K((\mathbb{R}^{\leq 0}))$. Assume that*

$$\mathrm{ot}(a) = \omega^2 \quad \textit{or} \quad \mathrm{ot}(a) = \omega^2 + 1,$$

and that 0 is an accumulation point of $\mathrm{supp}(a)$*, equivalently $a \notin J + K$. In $K((\mathbb{R}^{\leq 0}))/(J + K)$, take the K-linear span of the classes of the translated truncations $a^{|x}$ for $x < 0$. If this span has dimension greater than 2, then a is irreducible in $K((\mathbb{R}^{\leq 0}))$. This is [PS06, Cor. 3.3] (Theorem A).*

Proof. Suppose that $a = bc$. After exchanging the factors if necessary, the factorisation classification for these two support order types leaves two cases: either b is a nonzero constant, or $v_J(b) = v_J(c) = \omega$. In the second case, multiplicativity of v_J from Fact 32 (*Multiplicative property of the ordinal value (Ber00, Theorem 9.7)*), together with the support-tail bound for negative translated truncations, forces the critical points of b and c to be 0. The convolution formula modulo $J + K$ then writes, for every $x < 0$, the class of $a^{|x}$ as

$$c_x[b] + b_x[c].$$

Thus the classes of translated truncations of a lie in the span of $[b]$ and $[c]$, which has dimension at most 2, a contradiction. Hence one factor is a unit, and a is irreducible. □

`PommersheimShahriari.irreducible_of_two_lt_rank_translatedTruncationSpan`

Fact 129 (Greatest common divisors of finite-support series)**.** *Let G be a linearly ordered abelian group and K a field. For all p, q in the finite-support subring $K(G^{\leq 0})$ of $K((G^{\leq 0}))$, there is $d \in K(G^{\leq 0})$ such that, for every $e \in K(G^{\leq 0})$,*

$$e \mid p \textit{ and } e \mid q \quad \Longleftrightarrow \quad e \mid d.$$

This is the greatest-common-divisor assertion of [LM24, Fact 2.5.2].

Proof. The zero cases are immediate. Otherwise, the supports of p and q generate a finite-rank free subgroup H of G. Finite-support series with exponents in H form a Laurent polynomial ring, hence a unique factorisation domain. Compute a least common multiple there, compare coefficients on cosets of H to transfer divisibility to finite-support series with exponents in G, and use the lcm-to-gcd construction. Write each input as a monomial times a series whose support meets 0, and translate the resulting gcd by the larger of the two monomial exponents. The gcd and its two cofactors then have nonpositive support. ☐

`HahnSeries.Nonpositive.finiteSupport_pairwise_gcd_exists`

Theorem 130 (Greatest common divisors in $K((\mathbb{R}^{\leq 0}))$). *Let K be a field of characteristic 0. The ring $K((\mathbb{R}^{\leq 0}))$ of generalised power series with nonpositive real exponents is a GCD domain.*

Proof. By Lemma 2 (*Extension of a degreewise independent family*), choose a minimal homogeneous generating system of $\widehat{\mathrm{P}}$ together with principal-series representatives of its generators. By Theorem 118 (*Polynomial presentation of the series ring*), $K((\mathbb{R}^{\leq 0}))$ is a polynomial ring over the finite-support subring $K(\mathbb{R}^{\leq 0})$. By Fact 129 (*Greatest common divisors of finite-support series*), the coefficient ring is a GCD domain. Normalising each nonzero greatest common divisor by the coefficient of $t^{\sup(p)}$ supplies the hypothesis required by Lemma 10 (*Greatest common divisors in multivariate polynomial rings*). Transport the resulting gcd operation across the isomorphism. ☐

`Berarducci.nonemptyGCDMonoid`

Theorem 131 (Primality of generalised power series). *Let K be a field of characteristic 0. Every element of $K((\mathbb{R}^{\leq 0}))$ is primal in $K((\mathbb{R}^{\leq 0}))$.*

Proof. By Theorem 130 (*Greatest common divisors in $K((\mathbb{R}^{\leq 0}))$*), the series ring is a GCD domain. Every GCD domain is pre-Schreier [LM24, Fact 2.5.1], and every element of a pre-Schreier domain is primal [LM24, §2.5]. ☐

`Berarducci.isPrimal`

Corollary 132 (Primality criterion for series of degree one). *Let K be a field of characteristic 0 and $b \in K((\mathbb{R}^{\leq 0}))$ a series with $\deg(b) = 1$. Then b is prime in $K((\mathbb{R}^{\leq 0}))$ if and only if every divisor of b with finite support is a unit.*

Proof. Suppose first that b is prime and write $b = pq$ with p of finite support. Irreducibility makes p or q a unit. If q were a unit, multiplicativity of the degree would give $\deg(p) = 1$, contradicting the finite support of p; hence p is a unit. Conversely, assume that every finite-support divisor of b is a unit. In a factorisation $b = cd$ into nonzero factors, degree multiplicativity gives $\deg(c)+\deg(d) = 1$, so one factor has degree 0 and therefore finite support. The hypothesis makes that factor a unit, proving that b is irreducible. Every series is primal by Theorem 131 (*Primality of generalised power series*), so b is prime. ☐

`Berarducci.prime_iff_of_degree_eq_one`

Corollary 133 (Irreducible series are prime). *Let K be a field of characteristic 0. Every irreducible element of $K((\mathbb{R}^{\leq 0}))$ is prime in $K((\mathbb{R}^{\leq 0}))$.*

Proof. Every series is primal by Theorem 131 (*Primality of generalised power series*), and an irreducible primal element is prime. ☐

`Berarducci.prime_of_irreducible`

Corollary 134 (Random series of finite Cantor degree are prime)**.** *Let K be a field of characteristic* 0*. Let $m, n \geq 1$, let $\beta < \omega^n$, and let $b \in K((\mathbb{R}^{\leq 0}))$ be random. If*

$$\sup(b) = 0, \qquad \operatorname{ot}(\operatorname{supp}(b)) = \omega^n \cdot m + \beta,$$

then b is prime in $K((\mathbb{R}^{\leq 0}))$.

Proof. [FLLM, Theorem 1.8] makes b irreducible. By Corollary 133 (*Irreducible series are prime*), every irreducible series in $K((\mathbb{R}^{\leq 0}))$ is prime. ☐

`Berarducci.prime_of_isRandom`

Corollary 135 (Lower-order perturbations of random series are prime)**.** *Let K be a field of characteristic* 0*. Let $m, n \geq 1$ and $\beta < \omega^n$. Suppose that $b \in K((\mathbb{R}^{\leq 0}))$ is random and*

$$\operatorname{ot}(\operatorname{supp}(b)) = \omega^n \cdot m + \beta.$$

If $r \in K((\mathbb{R}^{\leq 0}))$ satisfies

$$\operatorname{ot}(\operatorname{supp}(r)) < \omega^n, \qquad \sup(b + r) = 0,$$

then $b + r$ is prime in $K((\mathbb{R}^{\leq 0}))$.

Proof. [FLLM, Theorem 1.8] makes $b+r$ irreducible. Apply Corollary 133 (*Irreducible series are prime*). ☐

`Berarducci.prime_add_of_isRandom`

Corollary 136 (Random principal series of positive finite degree are prime)**.** *Let K be a field of characteristic* 0*. Every random principal series $b \in K((\mathbb{R}^{\leq 0}))$ with*

$$0 < \deg(b) < \omega$$

is prime in $K((\mathbb{R}^{\leq 0}))$.

Proof. Write $\deg(b) = n$ with $1 \leq n < \omega$. [FLLM, Corollary 1.5] makes b irreducible, so Corollary 133 (*Irreducible series are prime*) makes it prime. ☐

`Berarducci.prime_of_isRandom_of_isPrincipal`

Corollary 137 (Unique factorisation under support-order conditions)**.** *Let K be a field of characteristic* 0 *and $a \in K((\mathbb{R}^{\leq 0}))$ a non-zero series. Suppose that $\operatorname{ot}(a) = v_J(a)$, or that $v_J(a) > 1$ and $\operatorname{ot}(a) = v_J(a) + 1$. Then a admits a factorisation into irreducibles, unique up to reordering and up to multiplication of the factors by non-zero elements of K. These are the two alternatives of [LM17, Definition 4.1].*

Proof. Under either support-order hypothesis, [LM17, Theorem 4.8] gives a finite factorisation into irreducibles. By Corollary 133 (*Irreducible series are prime*), every such irreducible is prime. Uniqueness of finite prime factorisations then shows that any two multisets of factors are related, after reordering, by association. ☐

`LM17.exists_unique_factorization_of_supportOrderType_eq_ordinalValue_or_add_one`

10 Reduction along Archimedean classes

The real-exponent theorem handles one Archimedean scale, but an exponent group satisfying assumptions $(A1)_\sigma$ and $(A2)_\sigma$ may contain many such scales. The proof separates the leading Archimedean class from the smaller ones. Regrouping the series treats the smaller exponents as coefficients in a larger field. Assumption $(A1)_\sigma$ supplies an order isomorphism $H_\sigma \cong \mathbb{R}$, so the result of the preceding chapter applies.

Assumption $(A2)_\sigma$ ensures that the corresponding κ-bounded integer part has the required fraction field. Convexity of the supports of factors then lets us carry primality back through the change of coordinates. For a reduced series, this makes the factor at the leading class primal. Removing that factor leaves only smaller Archimedean classes in the support. Repeating the step proves primality when the support meets finitely many Archimedean classes. The next chapter combines this finite result with a quotient argument that applies whenever the support meets infinitely many Archimedean classes.

Theorem 138 (Fraction fields of bounded Hahn integer parts)**.** *Let G be an ordered abelian group, let K be a field, and let $\kappa > \aleph_0$. If $\kappa \leq \mathrm{cof}(G)$, then the bounded Hahn field $K((G))_\kappa$ is the fraction field of $Z + K((G^{<0}))_\kappa$ for every subring $Z \subseteq K$.*

Proof. The support of a series in $K((G))_\kappa$ has cardinality less than κ, so it is not cofinal in G. Choose an upper bound x for the support and put $u = \max\{x, 0\}$. Multiplication by t^{-u} moves the support strictly below zero and gives constant coefficient zero. Thus both t^{-u} and $t^{-u}b$ belong to the bounded Hahn integer part, and $b = (t^{-u}b)/t^{-u}$. ☐

`HahnSeries.fracSubring_cardSuppLTTruncationIntegerPart_eq_top_of_le_cof`

Lemma 139 (Convexity of the supports of factors)**.** *Let H be a convex subgroup of a linearly ordered abelian group G. If $a, b \in K((G^{\leq 0}))$ are nonzero and* $\mathrm{supp}(ab) \subseteq H$*, then*

$$\mathrm{supp}(a) \subseteq H, \qquad \mathrm{supp}(b) \subseteq H.$$

Proof. The least support exponents satisfy $\min \mathrm{supp}(ab) = \min \mathrm{supp}(a) + \min \mathrm{supp}(b)$. This sum and zero lie in H, while both summands are nonpositive, so convexity puts each least exponent in H. Every later support exponent lies between its least exponent and zero and therefore also belongs to H. ☐

`HahnSeries.Nonpositive.support_subset_convex_of_mul_support_subset`

Fact 140 (Primality transfer at the leading Archimedean class)**.** *Let κ be an uncountable regular cardinal, K an Archimedean ordered division ring, G a linearly ordered abelian group and ordered K-module, R a field, and Z a subring of R; choose the Archimedean splitting of G. Let $b \in Z + R((G^{<0}))_\kappa$ be nonzero and reduced, with nonzero lowest exponent, and set $\sigma = [v(b)]$ and $L_\sigma = R((G_{\prec\sigma}))_\kappa$. If $(A2)_\sigma$ holds whenever the coefficient of exponent 0 in $\iota_\sigma(b)$ is zero, then b is primal in $Z + R((G^{<0}))_\kappa$ if and only if $\iota_\sigma(b)$ is primal in $L_\sigma((H_\sigma^{\leq 0}))$. This is the set-sized κ-bounded form of [LM24, Prop. 9.2.2].*

Proof. First restrict to the subring fixed by T_σ; this does not change the primality of b. The splitting ι_σ identifies that subring with $S_\sigma + L_\sigma((H_\sigma^{<0}))$. Reducedness says that the coefficient of exponent 0 is either 0 or 1. In the coefficient-one branch it is a unit, so the transfer lemma applies directly. In the coefficient-zero branch, the cofinal alternative in $(A2)_\sigma$ and Theorem 138 (*Fraction fields of bounded Hahn integer parts*) give $\mathrm{Frac}(S_\sigma) = L_\sigma$; the degenerate alternative gives the same equality directly. The transfer lemma therefore applies in both cases. Transport primality back through the splitting. Regularity makes the split and unsplit support bounds compatible, while uncountability keeps quotient supports below κ. □

`HahnSeries.Nonpositive.isPrimal_iff_isPrimal_splitTruncationCardSuppLT_of_isReduced_if_A2`

Corollary 141 (Primality of reduced bounded Hahn integer-part series)**.** *Let K be an Archimedean ordered division ring, G an ordered K-vector space, R a field of characteristic 0, $\kappa > \aleph_0$ a regular cardinal, and $Z \subseteq R$ a subring. For every nonzero Archimedean class τ of G, fix a complement H_τ of $G_{\prec\tau}$ in $G_{\preceq\tau}$, and write $L_\tau := R((G_{\prec\tau}))_\kappa$. Let $b \in Z + R((G^{<0}))_\kappa$ be reduced, with nonzero underlying series and nonzero order, and let σ be its leading Archimedean class. If $H_\sigma \simeq \mathbb{R}$ as ordered additive groups, and either $G_{\prec\sigma}$ has cofinality at least κ or $G_{\prec\sigma} = \{0\}$ and every element of R is a fraction of elements of Z, then b is primal in $Z + R((G^{<0}))_\kappa$.*

Proof. By Fact 140 (*Primality transfer at the leading Archimedean class*), primality of b is equivalent to primality of $\iota_\sigma(T_\sigma b)$ in $L_\sigma((H_\sigma^{\leq 0}))$. Transport the exponent group through $H_\sigma \simeq \mathbb{R}$. By Theorem 131 (*Primality of generalised power series*), the transported series is primal in $L_\sigma((\mathbb{R}^{\leq 0}))$, so the equivalence transfers primality back to b. □

`HahnSeries.Nonpositive.isPrimal_of_isReduced_of_leadingClass_orderIso_real`

Theorem 142 (Primality for finitely many Archimedean support classes)**.** *Let K be an Archimedean ordered division ring, G an ordered K-vector space, R a field of characteristic zero, $\kappa > \aleph_0$ a regular cardinal, and $Z \subseteq R$ a pre-Schreier subring. At every nonzero Archimedean class σ, choose an additive complement H_σ to $G_{\prec\sigma}$ in $G_{\preceq\sigma}$ that is order additively isomorphic to $\mathbb{R}$. Assume that $G_{\prec\sigma}$ either has cofinality at least κ, or is zero and every element of R is a fraction of elements of Z. Every element of $Z + R((G^{<0}))_\kappa$ whose support meets only finitely many Archimedean classes is primal.*

Proof. Induct on the number of Archimedean classes met by the support. At the leading class, split the series into its reduced factor and its strict lower-class factor. The reduced factor is primal by Corollary 141 (*Primality of reduced bounded Hahn integer-part series*); the other factor meets strictly fewer classes and is primal by induction. A product of primal elements is primal. □

`HahnSeries.Nonpositive.isPrimal_of_supportArchimedeanClasses_finite`

11 Refinement and transfinite induction

Finite induction is not enough when the set of Archimedean classes met by the support is infinite. Its well-order splits into a non-zero limit initial segment and a finite final segment. A common tail lies below every class in the limit segment, and the proof quotients the exponent group by this tail. If the quotient is Cauchy complete, the corresponding quotient ring of germs has a polynomial presentation. An equality of products can then be refined into four factors at a suitable Archimedean class of the quotient.

The proof normalises this refinement and turns the four congruences of germs into an exact refinement of the restrictions to the corresponding closed Archimedean ball in the integer-part ring. Factoring off the resulting divisor leaves a cofactor whose support meets Archimedean classes of strictly smaller order type. Combining this refinement with primality of that cofactor makes transfinite induction possible, including at infinite successor ranks. This extends the result for supports meeting finitely many Archimedean classes to all series allowed by the cardinal bound. It assumes completeness and a fraction-field property for every common tail; the remaining chapters prove those facts in the cases we need.

Lemma 143 (Normalization of a closed-class refinement in a Hahn integer part)**.** *Let $S \subseteq L$ be a subring of a field, let q be a nonzero Archimedean class of an ordered exponent group, and let $a, b, c, d \in S + L((G^{<0}))_\kappa$ satisfy $ab = cd$. Assume the constant coefficient of a is primal in S, every element of L is a fraction of elements of S, and the closed-class restriction a_q is nonzero.*

If the restrictions of a, b, c, d admit a four-factor refinement by κ-bounded nonpositive Hahn series, then they admit such a refinement by elements of $S + L((G^{<0}))_\kappa$.

Proof. Regard the bounded Hahn integer part as the inverse image of S under the constant-coefficient homomorphism on bounded nonpositive series. The primality of the constant coefficient of a and the fraction-field hypothesis allow a common scalar adjustment of the four ambient factors so that all four constant coefficients lie in S. Transport the adjusted refinement back through this residue-subring presentation. □

`HahnSeries.CardSuppLTTruncationIntegerPart.exists_refinement_closedClassRestrict_of_ambient`

Lemma 144 (Closed-ball restriction under quotient regrouping)**.** *Let P be a convex subspace of an ordered vector space G, and regroup a bounded Hahn series first by exponents in G/P and then by exponents in P. For an Archimedean class q of G/P, restriction of the regrouped series to the closed ball of q is the regrouping of the original series restricted to the inverse image of that ball in G.*

Proof. Compare the coefficient at an outer exponent $\bar{g} \in G/P$ and an inner exponent $p \in P$. If $\bar{g}$ lies in the closed ball of q, both sides have the coefficient of the unique exponent of G represented by $(\bar{g}, p)$; otherwise both coefficients are zero. □

`HahnSeries.CardSuppLTTruncationIntegerPart.convexQuotientSplit_filter_eq_closed_class_restrict`

Theorem 145 (Four-factor refinement in the germ ring over a Cauchy-complete exponent group)**.** *Under the hypotheses of Theorem 114 (*Polynomial presentation of the germ ring over a Cauchy-complete exponent group*), if $a, b, c, d \in K((G^{\leq 0}))$ and $ab = cd$, then there are $e, f, g, h \in K((G^{\leq 0}))$ such that, modulo series whose support is bounded strictly below zero,*

$$a = ef, \qquad b = gh, \qquad c = eg, \qquad d = fh.$$

Proof. By Theorem 114 (*Polynomial presentation of the germ ring over a Cauchy-complete exponent group*), the germ ring is a polynomial ring over K, hence is a unique factorisation domain and has the refinement property. Refine the four images in the quotient and choose series representing the four factors. □

`HahnSeries.Nonpositive.exists_germ_refinement_of_complete_exponent_group`

Lemma 146 (Divisibility from a convex restriction)**.** *Let C be a convex subgroup of an ordered abelian group G, and let $q, w, x \in Z + K((G^{<0}))_\kappa$. Suppose $q \neq 0$, $\operatorname{supp}(q) \subseteq C$, and*

$$x_{|C} = qw.$$

Then $q \mid x$ in $Z + K((G^{<0}))_\kappa$.

Proof. Divide x by q in the bounded Hahn field. On C the quotient is w. Outside C, convexity and nonpositivity keep every exponent of the quotient strictly negative; the constant coefficient is therefore the constant coefficient of w and lies in Z. Hence the quotient belongs to the bounded Hahn integer part. $\square$

`HahnSeries.CardSuppLTTruncationIntegerPart.dvd_of_restriction_factorization`

Lemma 147 (Transport of refinement from a quotient Archimedean class). *Let P be a convex subspace of an ordered vector space G, let q be a nonzero Archimedean class of G/P, and let $t \neq 0$ be the restriction of a to the inverse image of the closed ball of q. If the closed-ball restrictions of a, c, d in the iterated Hahn field factor as*

$$a_q = ef, \qquad c_q = eg, \qquad d_q = fh,$$

then there are $e_A, f_A \in Z + R((G^{<0}))_\kappa$ such that $t = e_A f_A$, $e_A \mid c$, and $f_A \mid d$.

Proof. By Lemma 144 (*Closed-ball restriction under quotient regrouping*), the Hahn-field isomorphism obtained by regrouping exponents along P identifies closed-ball restriction in G/P with restriction to its inverse image in G. Transport e and f back through this isomorphism. Since their product is supported in the retained convex subgroup, Lemma 139 (*Convexity of the supports of factors*) confines both supports there. Apply Lemma 146 (*Divisibility from a convex restriction*) to the restricted factorisations of c and d to obtain $e_A \mid c$ and $f_A \mid d$ in the ambient integer part. $\square$

`HahnSeries.CardSuppLTTruncationIntegerPart.exists_factor_refinement_of_closed_class_refinement`

Lemma 148 (Factorisation by a convex restriction). *Let C be a convex subgroup of an ordered abelian group G, and let $b \in Z + K((G^{<0}))_\kappa$ have nonzero restriction $t = b_{|C}$. Then*

$$b = tw$$

for some $w \in Z + K((G^{<0}))_\kappa$ with constant coefficient 1*. Every nonzero exponent in* $\operatorname{supp}(w)$ *lies outside C, and its Archimedean class is met by* $\operatorname{supp}(b)$.

Proof. Divide b by its nonzero restriction t in the bounded Hahn field. The quotient has constant coefficient 1. Convexity separates the discarded support from C; the Hahn inverse expansion shows that each class appearing in the quotient already appears in the support of b. Thus the quotient remains in the bounded Hahn integer part and has the stated support. $\square$

`HahnSeries.CardSuppLTTruncationIntegerPart.exists_factorization_by_restriction`

Lemma 149 (Support classes beyond a quotient Archimedean ball). *Let T be a family of nonzero Archimedean classes of an ordered rational vector space G, and let q be a nonzero Archimedean class of the quotient by the common tail below T. Suppose q is met by the image of a set $S \subseteq G$. If every nonzero element of $W \subseteq G$ maps outside the closed ball of q and has an Archimedean class met by S, then there is a nonzero class c met by S such that every class met by W is strictly below c and is met by S.*

Proof. Choose $y \in S$ whose image has class q. For $g \in W \setminus \{0\}$, being outside the closed ball of q says that the quotient class of g is strictly larger in magnitude than that of y. The common-tail quotient reflects this strict comparison to the original Archimedean-class order, so the class of g lies strictly below the class of y. $\square$

`HahnSeries.exists_nonzeroSupportClass_bound_tailQuotient`

Lemma 150 (An irreducible germ with zero constant coefficient)**.** *Let K be a field of characteristic zero and let G be a nontrivial, densely ordered abelian group with no least or greatest element and no least nonzero Archimedean magnitude. Assume that G is Cauchy complete for its additive uniformity. Then some series $p \in K((G^{\leq 0}))$ has constant coefficient zero and irreducible image in the germ ring at zero.*

Proof. Choose a well-ordered support cofinal at 0 that meets a cofinal family of nonzero Archimedean classes. The resulting series has a nonzero, nonunit germ. Hence Theorem 114 (*Polynomial presentation of the germ ring over a Cauchy-complete exponent group*) has a nonempty index set. Choose a polynomial variable X_i and a series p representing it. Subtracting the constant coefficient r of p gives a representative with constant coefficient zero and germ corresponding to $X_i - r$, which is irreducible. □

`HahnSeries.Nonpositive.exists_irreducible_cantorBendixson_germ_with_constantCoeff_zero`

Theorem 151 (Refinement forces $K = \mathrm{Frac}(Z)$)**.** *Let K be a field of characteristic zero and let G satisfy the hypotheses of Lemma 150 (*An irreducible germ with zero constant coefficient*). If*

$$Z + K((G^{<0}))$$

has the refinement property for a subring $Z \subseteq K$, then $K = \mathrm{Frac}(Z)$.

Proof. Identify the integer part with the inverse image of Z under the constant-coefficient map. By Lemma 150 (*An irreducible germ with zero constant coefficient*), it contains a series p whose constant coefficient is zero and whose germ is irreducible. Every representative of a unit germ has nonzero constant coefficient, and the refinement property makes p primal. The scalar argument in Lemma 1 (*A primal zero-residue element forces the fraction-field condition*) therefore gives $K = \mathrm{Frac}(Z)$. □

`HahnSeries.Nonpositive.fracSubring_eq_top_of_hasFourFactorRefinement_truncationIntegerPart`

Lemma 152 (Strict decrease of support-class order type)**.** *Let $x, y \in K((G^{\leq 0}))$. If c is a nonzero Archimedean class met by $\mathrm{supp}(x)$ and every nonzero class met by $\mathrm{supp}(y)$ is a class met by $\mathrm{supp}(x)$ and is strictly below c, then the order type of the nonzero support classes of y is strictly smaller than that of x.*

Proof. The classes met by either nonpositive support are partially well ordered. Monotonicity of order type embeds the classes of y into the initial segment of the classes of x below c, and a proper initial segment of a well-order has strictly smaller order type. □

`HahnSeries.Nonpositive.orderType_nonzeroSupportArchimedeanClasses_lt`

Lemma 153 (Factorisation at a quotient Archimedean class)**.** *Let T be a set of nonzero Archimedean classes of an ordered rational vector space G, and let q be a nonzero Archimedean class of the quotient of G by the common tail below T. Suppose the support of $b \in Z + R((G^{<0}))_\kappa$ meets q and the restriction t of b to the inverse image of the closed ball of q is nonzero. Then*

$$b = tw,$$

where the order type of the nonzero Archimedean support classes of w is strictly smaller than that of b.

Proof. By Lemma 148 (*Factorisation by a convex restriction*), the retained restriction gives a factorisation $b = tw$, and every nonzero class met by w is also met by b. The quotient class q and Lemma 149 (*Support classes beyond a quotient Archimedean ball*) bound all those classes strictly below one class met by b. The strict order-type inequality follows from Lemma 152 (*Strict decrease of support-class order type*). ☐

`HahnSeries.CardSuppLTTruncationIntegerPart.exists_factor_with_smaller_support_class_orderType`

Lemma 154 (Well-ordered subsets of closed rational spans)**.** *Let C be a densely ordered rational vector space with its order topology, let $\kappa > \aleph_0$, and let $S \subseteq C$ be nonempty with $\#S < \kappa$. Every well-ordered subset of the topological closure of* $\operatorname{span}_{\mathbb{Q}}(S)$ *has cardinality less than κ.*

Proof. For each nonmaximal point x of the well-ordered subset, let x^+ be its successor and choose a point of $\operatorname{span}_{\mathbb{Q}}(S)$ strictly between x and x^+. The resulting intervals are disjoint, so this choice is injective; the possible maximum accounts for one additional point. Hence the subset has cardinality at most $\#\operatorname{span}_{\mathbb{Q}}(S) + 1$. Finite rational linear combinations identify the latter span with a subset of $S \to_0 \mathbb{Q}$, whose cardinality is $\max\{\#S, \aleph_0\} < \kappa$. ☐

`Submodule.mk_lt_of_isPWO_topologicalClosure_span`

Theorem 155 (Cardinal-bounded refinement modulo series bounded away from zero)**.** *Let C be an ordered rational vector space that is Cauchy complete for its additive uniformity and whose nonzero Archimedean classes have no least element in the magnitude order. Let K be a field of characteristic zero and let $\kappa > \aleph_0$. If C has a positive coinitial subset of cardinality less than κ, then every equation $ab = cd$ among four κ-bounded series in $K((C^{\leq 0}))$ admits a four-factor refinement modulo series bounded strictly below zero whose four factors are also κ-bounded.*

Proof. Put the supports of a, b, c, d and the chosen positive coinitial set into one closed rational subspace C_0. The coinitial set remains positive and coinitial in C_0; hence C_0 is nontrivial, has no endpoints, and its nonzero Archimedean classes have no least element in the magnitude order. As a closed subspace of C, it is Cauchy complete for its additive uniformity, and its induced order topology and rational vector-space structure satisfy the remaining hypotheses of Theorem 145 (*Four-factor refinement in the germ ring over a Cauchy-complete exponent group*). Apply that theorem over C_0, and then map the four factors back to C. Each factor has well-ordered support in C_0. By Lemma 154 (*Well-ordered subsets of closed rational spans*), such a support has cardinality less than κ. ☐

`HahnSeries.Nonpositive.exists_cardinal_germ_refinement`

Theorem 156 (Exact refinement over a Cauchy-complete common-tail quotient)**.** *Let T be a nonempty set of nonzero Archimedean classes with no least member in the magnitude order, and let C be the quotient of the exponent group by the common tail below T. Assume that C is Cauchy complete for its additive uniformity and that T has cardinality less than κ. For every equation $ab = cd$ among four κ-bounded series in $K((C^{\leq 0}))$ and every set U of nonzero Archimedean classes of C coinitial in the magnitude order, there is $q \in U$ and a κ-bounded four-factor refinement whose four equations hold exactly after restriction to the closed ball of q.*

Proof. Positive representatives of the classes in T give a positive coinitial subset of C of cardinality less than κ. Apply Theorem 155 (*Cardinal-bounded refinement modulo series bounded away from zero*); each of its four equations has an error supported strictly below zero. Coinitiality of U

in the magnitude order supplies a class q smaller in magnitude than all four errors, so closed-ball restriction makes the four equations exact simultaneously. □

`HahnSeries.Nonpositive.exists_closed_class_refinement_of_complete_tail_quotient`

Theorem 157 (Refinement at a quotient Archimedean class). *Let $a, b, c, d \in Z + R((G^{<0}))_\kappa$ satisfy $ab = cd$. Suppose that the Archimedean classes met by the support of a are $T_0 \cup T_1$, where T_1 is finite, and T_0 contains a non-zero class and has no greatest element. Assume primality for every series whose support meets only finitely many Archimedean classes. Assume that the quotient by the common tail below T_0 is Cauchy complete for its additive uniformity, and that the κ-bounded Hahn field on the common tail is the fraction field of its corresponding κ-bounded integer part. Then some Archimedean class of the quotient met by the support of a admits an exact four-factor refinement of the restrictions of a, b, c, d to its closed Archimedean ball.*

Proof. Regroup each series as a series on the common-tail quotient. The support classes of a are cofinal there, so Theorem 156 (*Exact refinement over a Cauchy-complete common-tail quotient*) gives an exact refinement at one class met by that support. Primality of the inner constant term and its descent to the convex common tail use Lemma 139 (*Convexity of the supports of factors*): every nonzero factor of a series supported in that tail is supported there. This primality together with the common-tail fraction-field hypothesis supplies the hypotheses of Lemma 143 (*Normalization of a closed-class refinement in a Hahn integer part*), which normalises the four quotient factors inside the bounded Hahn integer part. □

`HahnSeries.CardSuppLTTruncationIntegerPart.exists_closed_class_refinement_at_support_class`

Theorem 158 (Transfinite extension of finite-class primality). *Let G be an ordered rational vector space, R a field of characteristic zero, $\kappa > \aleph_0$ a regular cardinal, and $Z \subseteq R$ a subring. Assume that every element of $Z + R((G^{<0}))_\kappa$ meeting only finitely many Archimedean classes is primal. For every nonempty set T of nonzero Archimedean classes with no least member in the magnitude order and cardinality less than κ, assume that the quotient by the common tail below T is Cauchy complete for its additive uniformity and that the bounded Hahn field on the common tail is the fraction field of its bounded Hahn integer part. Then every element of $Z + R((G^{<0}))_\kappa$ is primal.*

Proof. Induct on the order type of the nonzero Archimedean support classes. The finite case is the hypothesis. Otherwise split the class set into a limit initial segment and a finite final segment. By Theorem 157 (*Refinement at a quotient Archimedean class*), an equation $ad = bc$ has an exact refinement at a quotient class met by the support of a. By Lemma 153 (*Factorisation at a quotient Archimedean class*), factoring off the corresponding retained block leaves a cofactor with strictly smaller support-class order type. The induction hypothesis makes that cofactor primal, while Lemma 147 (*Transport of refinement from a quotient Archimedean class*) transports the quotient refinement to the ambient integer part. These two refinements prove that a is primal. □

`HahnSeries.CardSuppLTTruncationIntegerPart.isPrimal_of_finite_classes_and_limit_tail_conditions`

Theorem 159 (Primality under finite-class and common-tail hypotheses). *Let G be an ordered rational vector space, R a field of characteristic zero, $\kappa > \aleph_0$ a regular cardinal, and $Z \subseteq R$ a pre-Schreier subring. At every nonzero Archimedean class, choose an additive complement to the strict inner ball that is order additively isomorphic to $\mathbb{R}$. Assume that each strict inner ball either has cofinality at least κ, or is zero and every element of R is a fraction of elements of Z.*

For every nonempty set T of fewer than κ nonzero Archimedean classes having no least member in the magnitude order, let H_T be the rational subspace of exponents lying beyond every class in T. Assume that G/H_T is Cauchy complete for its additive uniformity and that the bounded Hahn field on H_T is the fraction field of its bounded Hahn integer part. Then every element of $Z + R((G^{<0}))_\kappa$ is primal.

Proof. By Theorem 142 (*Primality for finitely many Archimedean support classes*), assumptions $(A1)_\sigma$–$(A3)$ make every series meeting finitely many Archimedean classes primal. Apply Theorem 158 (*Transfinite extension of finite-class primality*) to extend this result to arbitrary support-class order type. ☐

`HahnSeries.CardSuppLTTruncationIntegerPart.isPrimal_of_finite_class_assumptions_and_limit_tail_conditions`

12 A cut criterion for Cauchy completeness

The transfinite theorem needs quotients by common tails to be Cauchy complete. In the applications, however, the available hypothesis is about order: two small separated families have an element between them. This chapter turns that ability to fill cuts into Cauchy completeness.

Start with a Cauchy filter and choose points in sets of successively smaller diameter. The chosen points give the lower and upper sides of a cut. A point filling the cut lies within twice each chosen scale, so the filter converges. The chapter also shows that a monotone surjection which reflects strict inequalities passes cut filling to the quotient. This lets the same criterion handle both saturated ordered groups and surreal numbers.

Lemma 160 (Cut filling under monotone order-reflecting surjections)**.** *Let $f\colon G \twoheadrightarrow C$ be a monotone surjection of preordered sets that reflects strict inequalities. If every cut between two I-indexed families in G can be filled, then the same is true in C.*

Proof. Choose preimages in G of the two families in C. Reflection of strict inequalities keeps the lifted families separated. Fill their cut in G, then apply f; monotonicity places the image between the two original families. ☐

`FillsCuts.of_surjective`

Lemma 161 (A cut criterion for Cauchy completeness)**.** *Let G be an ordered abelian group whose order topology is induced by a compatible additive uniformity, and let I be nonempty. Suppose there are elements $\varepsilon_i > 0$ such that, for every $c > 0$, some $i \in I$ satisfies*

$$\varepsilon_i + \varepsilon_i \le c.$$

Suppose also that whenever $L_i < R_j$ for all $i, j \in I$, there is $z \in G$ with $L_i \le z \le R_j$ for all i, j. Then G is Cauchy complete.

Proof. For a Cauchy filter and each i, choose a set of diameter less than ε_i and a point a_i in it. Any two chosen sets meet, so

$$a_i - \varepsilon_i < a_j + \varepsilon_j$$

for all i, j. Fill this cut by z. Every point of the i-th chosen set then lies within $\varepsilon_i + \varepsilon_i$ of z. Coinitiality of these doubled scales shows that the filter converges to z. ☐

`completeSpace_of_coinitial_of_forall_exists_mem_cut`

13 Bounded generalised-power-series integer parts

We can now apply the transfinite theorem to a saturated ordered rational vector space. Under the uncountable saturation hypothesis used here, every Archimedean stratum is isomorphic to the additive ordered group of real numbers. Saturation also gives the required cofinality for strict inner balls and common tails. By the cut criterion from the last chapter, each common-tail quotient is Cauchy complete. The cofinality of the tail also gives the required fraction-field property for bounded series on that tail.

This checks every hypothesis used by the reduction and the transfinite induction. If the coefficient subring is pre-Schreier, then the bounded generalised-power-series integer part is also pre-Schreier and has four-factor refinement. Taking integer constant coefficients gives the corresponding theorem for saturated Hahn integer parts. No facts about surreal numbers have been used yet. The last chapter proves the same needed properties for surreal exponents in a different way.

Lemma 162 (Archimedean strata of saturated ordered groups)**.** *Let $\kappa > \aleph_0$ and let G be a κ-saturated ordered rational vector space. Every Archimedean stratum in any Hahn splitting of G is order additively isomorphic to $\mathbb{R}$.*

Proof. Embed a stratum order additively into $\mathbb{R}$ and fix a positive element whose image is $\rho > 0$. For $y \in \mathbb{R}$, the rational multiples $q\rho < y$ and $q\rho > y$ define two countable subsets of G. Saturation supplies an element between them. It lies in the corresponding closed Archimedean ball; project it to the stratum using the lexicographic splitting of that ball. Comparison with every rational multiple forces the projected element to map to y. Thus the embedding is surjective. ☐

`_private.ConwayRefinement.Standalone.Mathlib.Support.HahnIntegerPartRefinementProof.0.`
`ConwayRefinement.Standalone.Hahn.HahnIntegerPartRefinement.stratum_orderAddEquiv_real_of_`
`isKappaSaturated`

Lemma 163 (Cauchy completeness of common-tail quotients)**.** *Let G be a κ-saturated ordered rational vector space. Let T be a nonempty set of fewer than κ nonzero Archimedean classes with no least member in the magnitude order, and let H_T consist of the exponents beyond every class in T. Then G/H_T is Cauchy complete for its additive uniformity.*

Proof. Positive representatives of the classes in T descend to a coinitial family in G/H_T, and rational halving gives arbitrarily small doubled scales. Every cut between two T-indexed families in G is filled by κ-saturation; by Lemma 160 (*Cut filling under monotone order-reflecting surjections*), the monotone quotient map, which reflects strict inequalities, transfers this cut-filling property to G/H_T. Apply Lemma 161 (*A cut criterion for Cauchy completeness*). ☐

`_private.ConwayRefinement.Standalone.Mathlib.Support.HahnIntegerPartRefinementProof.0.`
`ConwayRefinement.Standalone.Hahn.HahnIntegerPartRefinement.completeSpace_tailQuotient_of_`
`isKappaSaturated`

Lemma 164 (Cofinality of common tails in saturated ordered groups)**.** *Let $\kappa > \aleph_0$, let G be a κ-saturated ordered rational vector space, and let T be a nonempty set of fewer than κ nonzero Archimedean classes with no least member in the magnitude order. Then the common tail H_T has cofinality at least κ.*

Proof. If a set $S \subseteq H_T$ of cardinality less than κ were cofinal, use $S \cup \{0\}$ as the left side of a cut. On the right put all rational fractions of positive representatives of classes in T. Saturation fills the cut by a positive element of H_T lying strictly above every member of S, a contradiction. ☐

_private.ConwayRefinement.Standalone.Mathlib.Support.HahnIntegerPartRefinementProof.0.
ConwayRefinement.Standalone.Hahn.HahnIntegerPartRefinement.cofinal_commonTail_of_isKappaSaturated

Lemma 165 (Cofinality of Archimedean inner balls in saturated ordered groups)**.** *Let $\kappa > \aleph_0$ and let G be a κ-saturated ordered rational vector space. At every nonzero Archimedean class, the strict inner ball has cofinality at least κ.*

Proof. If a set S of cardinality less than κ were cofinal in the strict inner ball below the class of $a \neq 0$, place $S \cup \{0\}$ on the left of a cut and the elements $|a|/(n+1)$ on the right. Saturation gives a positive element still in the strict inner ball and strictly above S, a contradiction. □

_private.ConwayRefinement.Standalone.Mathlib.Support.HahnIntegerPartRefinementProof.0.
ConwayRefinement.Standalone.Hahn.HahnIntegerPartRefinement.le_cof_ball_of_isKappaSaturated

Lemma 166 (Common-tail conditions in saturated ordered groups)**.** *Let $\kappa > \aleph_0$, let G be a κ-saturated ordered rational vector space, and let Z be a subring of a field R. For every nonempty set T of fewer than κ nonzero Archimedean classes with no least member in the magnitude order, the quotient G/H_T is Cauchy complete in its additive uniformity, and the bounded Hahn field $R((H_T))_\kappa$ is the fraction field of $Z + R((H_T^{<0}))_\kappa$.*

Proof. Quotient completeness follows from Lemma 163 (*Cauchy completeness of common-tail quotients*). By Lemma 164 (*Cofinality of common tails in saturated ordered groups*), $\kappa \leq \mathrm{cof}(H_T)$; hence Theorem 138 (*Fraction fields of bounded Hahn integer parts*) gives the asserted fraction-field equality. □

_private.ConwayRefinement.Standalone.Mathlib.Support.HahnIntegerPartRefinementProof.0.
ConwayRefinement.Standalone.Hahn.HahnIntegerPartRefinement.limitTailConditions_of_isKappaSaturated

Theorem 167 (Pre-Schreier property of bounded Hahn integer parts)**.** *Let G be an ordered rational vector space, R a field of characteristic zero, $\kappa > \aleph_0$ a regular cardinal, and $Z \subseteq R$ a pre-Schreier subring. Choose an additive complement to the strict inner ball at every nonzero Archimedean class. Assume every complement is order additively isomorphic to $\mathbb{R}$, and that at each such class either the strict inner ball has cofinality at least κ, or it is zero and every element of R is a fraction of elements of Z.*

For every nonempty set T of fewer than κ nonzero Archimedean classes having no least member in the magnitude order, let H_T be the rational subspace of exponents lying beyond every class in T. Assume that G/H_T is Cauchy complete for its additive uniformity and that the bounded Hahn field on H_T is the fraction field of its bounded Hahn integer part. Then $Z + R((G^{<0}))_\kappa$ is pre-Schreier.

Proof. The real-complement and inner-ball hypotheses are LM24 conditions ($A1$) and ($A2$), while the pre-Schreier hypothesis on Z is condition ($A3$). By Theorem 159 (*Primality under finite-class and common-tail hypotheses*), every element of the bounded Hahn integer part is primal. This is exactly the pre-Schreier property. □

ConwayRefinement.Standalone.Hahn.HahnIntegerPartRefinement.decompositionMonoid_of_assumptions

Corollary 168 (Pre-Schreier Hahn integer parts over saturated exponent groups)**.** *Let $\kappa > \aleph_0$ be regular, let G be a κ-saturated ordered rational vector space, let R be a field of characteristic zero, and let $Z \subseteq R$ be pre-Schreier. Then $Z + R((G^{<0}))_\kappa$ is pre-Schreier.*

Proof. Saturation gives real Archimedean strata by Lemma 162 (*Archimedean strata of saturated ordered groups*), the cofinal alternative in condition ($A2$) by Lemma 165 (*Cofinality of Archimedean inner balls in saturated ordered groups*), and the completeness and fraction-field conditions at common tails by Lemma 166 (*Common-tail conditions in saturated ordered groups*). These are the hypotheses of Theorem 167 (*Pre-Schreier property of bounded Hahn integer parts*). □

`ConwayRefinement.Standalone.Hahn.HahnIntegerPartRefinement.decompositionMonoid_of_saturation`

Theorem 169 (Refinement of bounded generalised-power-series integer parts). *Under the hypotheses of Theorem 167 (*Pre-Schreier property of bounded Hahn integer parts*), every equality $ab = cd$ in $Z + R((G^{<0}))_\kappa$ admits elements e, f, g, h in the same ring such that*

$$a = ef, \qquad b = gh, \qquad c = eg, \qquad d = fh.$$

Proof. By Theorem 167 (*Pre-Schreier property of bounded Hahn integer parts*), the bounded generalised-power-series integer part is pre-Schreier. The standard equivalence between primality of every element and four-factor refinement gives the displayed factors, which are then viewed as elements of the ambient generalised-power-series field. □

`ConwayRefinement.Standalone.Hahn.HahnIntegerPartRefinement.of_assumptions`

Corollary 170 (Refinement over saturated exponent groups). *Let $\kappa > \aleph_0$ be regular, let G be a κ-saturated ordered rational vector space, let R be a field of characteristic zero, and let $Z \subseteq R$ be pre-Schreier. Then every equality $ab = cd$ in $Z + R((G^{<0}))_\kappa$ has a refinement*

$$a = ef, \qquad b = gh, \qquad c = eg, \qquad d = fh$$

in the same ring.

Proof. By Lemma 162 (*Archimedean strata of saturated ordered groups*), saturation supplies condition ($A1$). By Lemma 165 (*Cofinality of Archimedean inner balls in saturated ordered groups*), it also supplies the cofinal alternative in condition ($A2$) at every nonzero Archimedean class. For each limit family, Lemma 166 (*Common-tail conditions in saturated ordered groups*) supplies quotient completeness and says that every bounded common-tail series is a fraction of two elements of the bounded common-tail Hahn integer part. These are all the hypotheses of Theorem 169 (*Refinement of bounded generalised-power-series integer parts*). □

`ConwayRefinement.Standalone.Hahn.HahnIntegerPartRefinement.of_saturation`

Corollary 171 (Refinement of saturated Hahn integer parts with integer constants). *Let $\kappa > \aleph_0$ be regular, let G be a κ-saturated ordered rational vector space, and let R be a field of characteristic zero. Then every equality $ab = cd$ in $\mathbb{Z} + R((G^{<0}))_\kappa$ has a refinement*

$$a = ef, \qquad b = gh, \qquad c = eg, \qquad d = fh$$

in the same ring.

Proof. The image of $\mathbb{Z}$ in R is pre-Schreier. Apply Corollary 170 (*Refinement over saturated exponent groups*) with this coefficient ring and rewrite membership as the requirement that the constant coefficient be an integer. □

`ConwayRefinement.Standalone.Hahn.HahnIntegerPartRefinement.of_saturation_integer_coefficients`

14 Surreal numbers and omnific integers

The last step is to connect the general series theorem to omnific integers and check its hypotheses for surreal exponents. Conway normal form gives a ring isomorphism after we negate the exponents. Under this isomorphism, omnific integers become bounded series with non-positive surreal exponents and an integer constant term. We can therefore solve the algebraic problem for series and carry the answer back.

Every nonzero surreal Archimedean stratum is isomorphic to the ordered additive group of real numbers, as required by the reduction to real exponents. Conway's cut construction gives the required cofinality statements and fills small cuts. The cut criterion then makes the common-tail quotients Cauchy complete, while cofinality gives the fraction-field property. The transfinite theorem shows that every element of the corresponding integer part is primal and hence gives four-factor refinement. Conway normal form carries this refinement to the omnific-integer subring, and the equivalence with Conway's cut definition gives the final theorem.

Theorem 172 (Primality of an explicit degree-two series)**.** *Let K be a field of characteristic 0. The series*

$$1+\sum_{m,n\in\mathbb{N}} t^{-\frac{1}{m+1}-\frac{1}{(m+1)(m+2)(n+1)}}$$

is prime in $K((\mathbb{R}^{\leq 0}))$.

Proof. The nonconstant support consists of ω successive blocks of order type ω and accumulates at 0; adjoining the constant term gives support order type ω^2+1. Three translated-truncation classes from distinct blocks are linearly independent modulo $J+K$, so their span has dimension greater than 2. Hence Fact 128 (*Irreducibility from translated truncation dimension*) makes the displayed series irreducible. It is prime by Corollary 133 (*Irreducible series are prime*). ☐

`Surreal.OmnificInteger.DegreeTwoExample.degreeTwoWithConstant_prime`

Theorem 173 (Equivalence of the cut and subring formulations of Conway's refinement conjecture)**.** *Conway's refinement conjecture for cut-defined omnific integers is equivalent to the refinement property of the omnific-integer subring of the surreal numbers.*

Proof. The cut predicate for omnific integers is equivalent to membership in the omnific-integer subring. Substitute this equivalence into the two four-factor statements; their equations and quantifiers are identical. ☐

`ConwayRefinement.Standalone.Oz.conwayConjecture_iff_native`

Fact 174 (Real structure of surreal Archimedean strata)**.** *Every nonzero Archimedean stratum of the surreal numbers is noncanonically isomorphic to* $(\mathbb{R},+,<)$ *as an ordered additive group [LM24, Proposition 2.4.3].*

Proof. Choose a positive element a of the stratum and send x to the standard part of x/a. Division by a makes the quotient finite; the chosen complement to the lower Archimedean ball makes the map injective, and the real multiples of a make it surjective. ☐

`Surreal.stratumOrderAddMonoidIsoReal`

Fact 175 (Cofinality of surreal Archimedean balls)**.** *Fix a universe u, let κ_u be its universe cardinal, and let c be a nonzero Archimedean class of $\mathbf{No}_u$. The strict Archimedean ball below c has cofinality at least κ_u [LM24, Proposition 2.4.4].*

Proof. A cofinal subset of cardinality below κ_u is u-small. Conway's cut construction gives a surreal number above that subset but still below every positive rational multiple of a representative of c, contradicting cofinality. $\square$

`Surreal.smallSupportCardinal_le_ball_cof`

Lemma 176 (Cofinality of common surreal Archimedean tails)**.** *Fix a universe u, let κ_u be its universe cardinal, and let T be a u-small family of nonzero Archimedean classes with no least member in the magnitude order. The common tail below T has cofinality at least κ_u.*

Proof. If a u-small subset were cofinal in the common tail, form a Conway cut above it and below the positive rational fractions of representatives of the classes in T. The resulting surreal belongs to the common tail but is above the proposed cofinal subset. $\square$

`Surreal.smallSupportCardinal_le_tailSubmodule_cof`

Theorem 177 (Fraction fields of surreal common-tail integer parts)**.** *Let T be a u-small family of nonzero Archimedean classes of $\mathbf{No}_u$ with no least member in the magnitude order, and let H_T be the common tail below T. For every field R and subring $Z \subseteq R$, the bounded Hahn field $R((H_T))_{\kappa_u}$ is the fraction field of $Z + R((H_T^{<0}))_{\kappa_u}$.*

Proof. By Lemma 176 (*Cofinality of common surreal Archimedean tails*), $\kappa_u \leq \mathrm{cof}(H_T)$. Therefore Theorem 138 (*Fraction fields of bounded Hahn integer parts*) applies to H_T. $\square$

`Surreal.fracSubring_cardSuppLTTruncationIntegerPart_tailSubmodule_eq_top`

Theorem 178 (The simplicity theorem for small cuts)**.** *Let $L, R \subseteq \mathbf{No}_u$ be small sets. If $l < r$ for every $l \in L$ and $r \in R$, then there is $x \in \mathbf{No}_u$ such that*

$$l < x < r$$

for every $l \in L$ and $r \in R$.

Proof. Form the surreal $\{L \mid R\}$. The separation hypothesis makes this a valid Conway cut, and the simplicity theorem places it strictly above every left option and strictly below every right option. Smallness keeps both option sets in the same universe. $\square$

`Surreal.exists_between_small_sets`

Theorem 179 (The signed normal-form isomorphism for omnific integers)**.** *For every universe u, signed Conway normal form restricts to a ring isomorphism*

$$\mathbf{Oz}_u \simeq \mathbb{Z} + \mathbb{R}((\mathbf{No}_u^{<0}))_{\kappa_u},$$

where κ_u is the universe cardinal.

Proof. Restrict Conway normal form to the omnific integers, then negate the exponents so that nonnegative Conway exponents become nonpositive exponents in the variable $t = \omega^{-1}$. $\square$

`Surreal.OmnificInteger.signedSmallSupportIntegerPartRingEquiv`

Corollary 180 (Primality for finitely many Archimedean classes)**.** *Every omnific integer whose normal-form exponents meet only finitely many Archimedean classes is primal in* **Oz***.*

Proof. Pass to the signed Hahn presentation and induct on the finite set of occupied Archimedean classes. A series of order zero is an integer constant. Otherwise, split off its leading class. The reduced factor is primal by Corollary 141 (*Primality of reduced bounded Hahn integer-part series*), using Fact 174 (*Real structure of surreal Archimedean strata*) for $(A1)_\sigma$ and Fact 175 (*Cofinality of surreal Archimedean balls*) for $(A2)_\sigma$; the lower truncation has fewer occupied classes and is primal by induction. Their product is primal. By Theorem 179 (*The signed normal-form isomorphism for omnific integers*), signed normal form transfers primality back to the original omnific integer. □

`Surreal.OmnificInteger.isPrimal_of_supportArchimedeanClasses_finite`

Theorem 181 (Primality of reduced omnific integers outside $\mathbb{Z}$)**.** *Let $x \in$ **Oz** be an omnific integer whose underlying surreal number is not equal to any integer. Write s for its signed nonpositive series representation. Suppose that s is reduced: $s \neq 0$ and, for some Archimedean class σ,*

$$\operatorname{supp}(s) \cap \operatorname{supp}(s-1) \subseteq \{y : [y] = \sigma\}.$$

Then x is primal in **Oz***.*

Proof. By Theorem 179 (*The signed normal-form isomorphism for omnific integers*), x corresponds to a bounded integer-part series b. Since x is not an ordinary integer, the underlying series of b is nonzero and has nonzero order; reducedness transfers from s to b. At the leading Archimedean class of b, the surreal exponent group satisfies $(A2)_\sigma$ by Fact 175 (*Cofinality of surreal Archimedean balls*), and its stratum is order-isomorphic to $\mathbb{R}$ by Fact 174 (*Real structure of surreal Archimedean strata*). Hence Corollary 141 (*Primality of reduced bounded Hahn integer-part series*) makes b primal. Transport primality back through the ring equivalence. □

`Surreal.OmnificInteger.isPrimal_of_isReduced`

Theorem 182 (Irreducible omnific integers are prime)**.** *Every irreducible element of* **Oz** *is prime in* **Oz***.*

Proof. Let $x \in$ **Oz** be irreducible. If x is not an ordinary integer, the factorisation at its leading Archimedean class shows that either x or $-x$ is reduced. The reduced element is primal by Theorem 181 (*Primality of reduced omnific integers outside* $\mathbb{Z}$); multiplication by -1 preserves primality, so x is primal and hence prime. If x is an ordinary integer, its irreducibility in **Oz** makes the corresponding integer irreducible in $\mathbb{Z}$, hence a prime integer. Divisibility of integer constants in **Oz** is detected by the ordinary integer constant coefficient, so x is prime in **Oz** in this case as well. □

`Surreal.OmnificInteger.prime_of_irreducible`

Theorem 183 (Primality of an explicit degree-two omnific integer)**.** *The omnific integer*

$$1 + \sum_{m,n\in\mathbb{N}} \omega^{\frac{1}{m+1} + \frac{1}{(m+1)(m+2)(n+1)}}$$

is prime in **Oz***.*

Proof. In the signed orientation $t = \omega^{-1}$, the nonconstant part is the real-exponent series above, now over the coefficient field of the leading Archimedean class. By Fact 128 (*Irreducibility from translated truncation dimension*), that series is irreducible. Coefficient transport and the residue-one irreducibility transfer then make the displayed omnific integer irreducible. It is prime by Theorem 182 (*Irreducible omnific integers are prime*). □

`Surreal.OmnificInteger.DegreeTwoExample.degreeTwoOz_prime`

Lemma 184 (A coinitial family of positive elements in surreal common-tail quotients)**.** *Let T be a u-small family of nonzero Archimedean classes of $\mathbf{No}_u$ with no least member in the magnitude order. Index T by a u-small type I, and let ε_i be the image in $\mathbf{No}_u/H_T$ of the positive representative of the class indexed by i. Then every ε_i is positive, and for every $x > 0$ in the quotient there is $i \in I$ such that $\varepsilon_i \leq x$.*

Proof. A positive representative cannot lie in H_T: a later class in T excludes it from the common tail. Conversely, lift $x > 0$ to a positive surreal representative. Since that representative is not in H_T, its Archimedean class lies above some member of T; the corresponding positive representative therefore maps below x in the quotient. □

`Surreal.rationalTailQuotientScale_pos_and_coinitial`

Lemma 185 (Cauchy completeness of surreal common-tail quotients)**.** *Let T be a u-small nonempty family of nonzero Archimedean classes of $\mathbf{No}_u$ with no least member in the magnitude order. The ordered rational vector space obtained by quotienting $\mathbf{No}_u$ by the common tail below T is Cauchy complete in its additive uniformity.*

Proof. By Lemma 184 (*A coinitial family of positive elements in surreal common-tail quotients*), a small copy of T indexes positive scales coinitial in the quotient, and rational halving supplies the doubled-scale hypothesis of Lemma 161 (*A cut criterion for Cauchy completeness*). To fill a cut in the quotient, Lemma 160 (*Cut filling under monotone order-reflecting surjections*) lifts its two small indexed families to surreal representatives. Strict order in a convex quotient reflects to those representatives, so Theorem 178 (*The simplicity theorem for small cuts*) fills the lifted cut inside $\mathbf{No}_u$; the monotone quotient map sends the filler back between the original families. The Cauchy-completeness criterion now applies. □

`Surreal.completeSpace_rationalTailQuotient`

Theorem 186 (Primality of the surreal Hahn integer part)**.** *Every element of*

$$\mathbb{Z} + \mathbb{R}((\mathbf{No}_u^{<0}))_{\kappa_u}$$

is primal, where κ_u is the universe cardinal.

Proof. The two Archimedean hypotheses follow from Fact 174 (*Real structure of surreal Archimedean strata*) and Fact 175 (*Cofinality of surreal Archimedean balls*). Therefore Theorem 142 (*Primality for finitely many Archimedean support classes*) handles series whose support meets finitely many Archimedean classes. For a small limit family, the quotient by its common tail is Cauchy complete by Lemma 185 (*Cauchy completeness of surreal common-tail quotients*). The common tail has the required fraction-field property by Theorem 177 (*Fraction fields of surreal common-tail integer parts*). Finally, Theorem 158 (*Transfinite extension of finite-class primality*) proves primality for arbitrary support-class order type. □

Theorem 187 (Refinement property of $\mathbf{Oz}_u$)**.** *If $a, b, c, d \in \mathbf{Oz}$ and $ab = cd$, then there are $e, f, g, h \in \mathbf{Oz}$ such that*

$$a = ef, \qquad b = gh, \qquad c = eg, \qquad d = fh.$$

Proof. By Theorem 179 (*The signed normal-form isomorphism for omnific integers*), $\mathbf{Oz}_u$ is isomorphic to its bounded generalised-power-series integer part. By Theorem 186 (*Primality of the surreal Hahn integer part*), that integer part is pre-Schreier and therefore has four-factor refinement. Transport the refinement back through signed normal form. □

Surreal.OmnificInteger.conwayRefinement

Theorem 188 (Conway's refinement theorem for omnific integers)**.** *Let a, b, c, d be surreal numbers satisfying Conway's cut equation $x = \{x - 1 \mid x + 1\}$. If $ab = cd$, then there are surreal numbers e, f, g, h, each satisfying the same cut equation, such that*

$$a = ef, \qquad b = gh, \qquad c = eg, \qquad d = fh.$$

Proof. By Theorem 173 (*Equivalence of the cut and subring formulations of Conway's refinement conjecture*), Conway's cut-defined statement is equivalent to the refinement property of the omnific-integer subring. Apply Theorem 187 (*Refinement property of $\mathbf{Oz}_u$*). □

ConwayRefinement.Standalone.Oz.ConwayConjecture.proof

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